2D Peskin Problems of an Immersed Elastic Filament in Stokes Flow

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Dedication

For Thomas, my mother, and my father – I love you more.

Abstract

In the work that follows we investigate a class of problems where a one dimensional closed elastic structure is immersed in a plane of steady Stokes flow. The dynamics are governed by a boundary integral equation describing the configuration of the immersed structure. Depending on the choice of elasticity law, we break our class into either a semilinear or fully nonlinear system of equations. In the nonlinear setting, we prove that the linearization of the system generates an analytic semigroup and use it to prove local existence and uniqueness in low regularity Hölder spaces. In the semilinear setting, we remove the principle operator via small scale decomposition and use it to build similar local existence results. Further, we establish spatial smoothness of solutions by careful estimates on a class of commutators. Using these regularity results, we are able to establish that the only equilibria of the system are uniformly parameterized circles which we then prove nonlinear stability about. Finally, we identify a quantity which classifies global-in-time behavior.

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Chapter 1

Introduction

Fluid structure interaction problems (FSI) form an extensive area of research within the sciences. They permeate a variety of different fields, ranging from airfoils in aerospace engineering to the movement of micro bacteria in biology and more. Their prevalence necessitates a rigorous mathematical treatment. We are interested in a particular class of fluid structure interaction problems wherein a closed one dimensional elastic structure partitions a two dimensional field of viscous fluid and is free to move with the fluid velocity. Our problem is inspired by the numerical method introduced by Peskin, the immersed boundary (IB) method [26, 27]. To honor his work, we have named this class of problems *Peskin Problems*.

Let Γ be a simple closed curve which partitions $\mathbb{R}^2$ into two regions, the interior of the curve, Ω_i and the exterior $\Omega_e = \mathbb{R}^2 \setminus \Omega_i$. Let Γ be parameterized by vector valued function $\boldsymbol{X}(\theta, t)$, where $\theta \in \mathbb{S}^1$ and $t \geq 0$. Here θ is the material coordinate, so that for fixed θ, $\boldsymbol{X}(\theta, t)$ moves with the local fluid velocity. Suppose further that the elastic structure has force density given by $\boldsymbol{F}(\boldsymbol{X}(\theta), t)$ which is of the form

$$\boldsymbol{F}(\boldsymbol{X}) = \partial_\theta \left(T \left(|\partial_\theta \boldsymbol{X}| \right) \boldsymbol{\tau}(\boldsymbol{X}) \right), \tag{1.1}$$

where T is the tension and $\tau = \partial_\theta \boldsymbol{X} / |\partial_\theta \boldsymbol{X}|$ is the unit tangent of the boundary Γ. Then, the system is governed by the following equations:

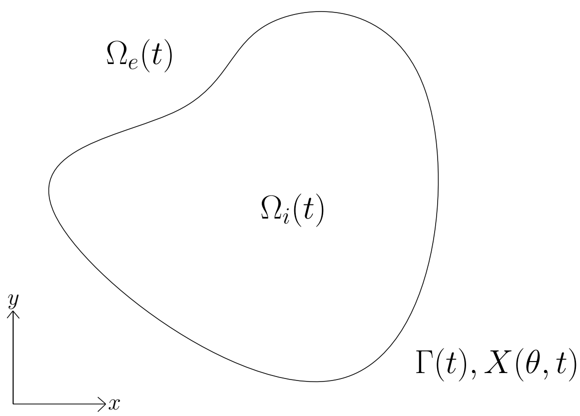

Figure 1.1: A closed curve partitions the plane

$$\mu \Delta \boldsymbol{u} - \nabla p = 0 \text{ in } \mathbb{R}^2 \backslash \Gamma, \tag{1.2}$$

$$\nabla \cdot \boldsymbol{u} = 0 \text{ in } \mathbb{R}^2 \backslash \Gamma, \tag{1.3}$$

$$[\![\boldsymbol{u}]\!] = 0 \text{ on } \Gamma, \tag{1.4}$$

$$[\![(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}} - pI)\boldsymbol{n}]\!] = \boldsymbol{F}(\boldsymbol{X}(\theta, t)) \left|\partial_\theta \boldsymbol{X}\right|^{-1} \text{ on } \Gamma, \tag{1.5}$$

$$\partial_t \boldsymbol{X} = \boldsymbol{u}(\boldsymbol{X}, t). \tag{1.6}$$

Equations (1.2) and (1.3) state that the fluid should satisfy the incompressible Stokes equation inside Ω_i and Ω_e with the viscosities of the fluid in the two domains equal.

For any quantity of interest w defined on $\Omega \backslash \Gamma$, let $[\![w]\!]$ denotes the jump in its value across Γ:

$$[\![w]\!] = w|_{\Gamma_{\mathrm{i}}} - w|_{\Gamma_{\mathrm{e}}} \tag{1.7}$$

where $w|_{\Gamma_{\mathrm{i}}}$ denotes the limiting value of w evaluated on Γ from the Ω_{i} side, and likewise for $w|_{\Gamma_{\mathrm{e}}}$. Using this notation, equation (1.4) states the that the fluid velocity should be continuous across the boundary Γ. Given this continuity, we are able to evaluate the velocity on Γ, so that equation (1.6) dictates that the structure move with the local fluid velocity.

Finally, equation (1.5) gives the jump in stress across the boundary. Here, $\nabla \boldsymbol{u}$ is the rate of deformation tensor and $(\nabla \boldsymbol{u})^T$, its transpose. Also, I is the identity matrix, and $\boldsymbol{n}$ is the outward unit normal on Γ (pointing from Ω_{i} to Ω_{e}). The notation $\partial_\theta \boldsymbol{X}$ denotes the derivative of

$\boldsymbol{X}$ with respect to θ, and $|\cdot|$ is the Euclidean length. The Jacobian factor $|\partial_\theta \boldsymbol{X}|$ is included to facilitate the change from Lagrangian to arclength coordinates. We must also impose boundary conditions on $\boldsymbol{u}$ and p as $\boldsymbol{x} \to \infty$. For this, we set

$$\boldsymbol{u} \to 0 \text{ as } |\boldsymbol{x}| \to \infty, \ p \text{ is bounded.} \tag{1.8}$$

This problem may be reformulated in several different ways. The *immersed boundary (IB)* reformulation of the system is to couple the fluid equations to the interfacial conditions by use of dirac delta. In this case, equations (1.2)-(1.5) are replaced with:

$$-\Delta \boldsymbol{u} + \nabla p = \int_{\mathbb{S}^1} \boldsymbol{F}(\boldsymbol{X}(\theta,t))\delta(\boldsymbol{x} - \boldsymbol{X}(\theta,t))d\theta, \quad \nabla \cdot \boldsymbol{u} = 0, \tag{1.9}$$

where $\delta(\boldsymbol{x} - \boldsymbol{X})$ is the Dirac delta function located at the point $\boldsymbol{X}$. The above equations are to be satisfied in a distributional sense in $\mathbb{R}^2$. Note that the interfacial conditions are now expressed in the form of a distributional body force on the right hand side of the Stokes equation. All other equations remain the same. We shall refer to the use of (1.2)-(1.5) as the *jump* formulation of the problem. If the functions $\boldsymbol{u}, p$ and $\boldsymbol{X}$ are sufficiently smooth, one can show that the two formulations are equivalent [17].

The IB formulation serves as the basis of the *immersed boundary (IB)* method. In the immersed boundary method, the fluid domain and the immersed elastic structure are discretized independently of each other, and communication between the two takes place solely through equations (1.6) and (1.9). The ease of implementation and robustness of the algorithm have enabled the simulation of challenging FSI problems and have made the IB method among the most popular numerical methods for FSI problems. We refer the reader to the review articles [23, 27] for details.

Finally, we consider the *boundary integral (BI)* formulation of the Peskin problem. Given the linearity of Stokes equation, solutions can be written as a convolution against the fundamental solution. Equations (1.2)-(1.5), together with condition (1.8), can be used to solve for $\boldsymbol{u}$ and

p to yield:

$$\boldsymbol{u}(\boldsymbol{x},t) = \int_{\mathbb{S}^1} G(\boldsymbol{x} - \boldsymbol{X}(\theta',t))\boldsymbol{F}(\boldsymbol{X}(\theta',t))d\theta', \tag{1.10}$$

$$\begin{aligned} G(\boldsymbol{x}) &= \frac{1}{4\pi}\left(-\log|\boldsymbol{x}|\,I + \frac{\boldsymbol{x}\otimes\boldsymbol{x}}{|\boldsymbol{x}|^2}\right) \\ &= \frac{1}{4\pi}\left(-\log|\boldsymbol{x}|\,I + \frac{1}{|\boldsymbol{x}|^2}\begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}\right),\ \boldsymbol{x} = (x,y)^{\mathrm{T}}, \end{aligned} \tag{1.11}$$

$$p(\boldsymbol{x},t) = \int_{\mathbb{S}^1} \boldsymbol{P}_{\mathrm{st}}(\boldsymbol{x} - \boldsymbol{X}(\theta',t)) \cdot \boldsymbol{F}(\boldsymbol{X}(\theta',t))d\theta', \quad \boldsymbol{P}_{\mathrm{st}}(\boldsymbol{x}) = \frac{\boldsymbol{x}}{2\pi|\boldsymbol{x}|^2}. \tag{1.12}$$

Here, G is the Stokeslet, the fundamental solution of the Stokes equation in $\mathbb{R}^2$. We note here that we will not suffer from the Stokes paradox of logarithmic growth of the velocity field $\boldsymbol{u}$ at infinity provided that the integral of $\boldsymbol{F}(\boldsymbol{X})$ over θ vanishes. As assumed above, $\boldsymbol{F}$ is a perfect derivative and so this is therefore the case. Given the continuity of the velocity $\boldsymbol{u}$ across Γ and the fact that the fluid viscosities in the two regions agree, we may limit in to evaluate equation (1.10) on Γ. Thus, we obtain the following closed equation for the evolution of $\boldsymbol{X}$.

$$\partial_t \boldsymbol{X}(\theta,t) = \int_{\mathbb{S}^1} G(\boldsymbol{X} - \boldsymbol{X}')\boldsymbol{F}(\boldsymbol{X}(\theta',t))d\theta'. \tag{1.13}$$

In the above and henceforth we write $\boldsymbol{X}' = \boldsymbol{X}(\theta',t)$, and we use similar notation for other primed quantities. From the BI formulation, it is clear that the only initial condition that needs to be supplied to this problem is the initial configuration $\boldsymbol{X}(\theta,0) = \boldsymbol{X}_0(\theta)$.

These three formulations of the problem can be shown to be equivalent provided that $\boldsymbol{u}$, $\boldsymbol{X}$, and p are sufficiently smooth. One problem formulation may be preferable to work with over the other to obtain certain estimates or quantities. Indeed, for the majority of our work, we will focus on a variant of the BI formulation but will switch to the jump formulation for the calculation of equilibria as a simple argument exists there.

All of the three formulations are the basis of computational methods for this problem. The jump formulation is used in the immersed interface method [19, 18] and moving mesh methods such as the Arbitrary Lagrangian Eulerian (ALE) method [9], the IB formulation in the immersed boundary and related methods [27, 23, 37, 32, 13] and the BI formulation can be used as a starting point for a boundary element/collocation method [28, 13]. Establishing sufficient smoothness of the solution, therefore, is important from both analytic and numerical points of view. From a numerical standpoint, unless some smoothness is established, it may not be clear whether the various methods are approximating the same solution. The wealth of numerical methods that can be used to tackle this problem has the potential to make the Peskin problems a standard testbed for the numerical analysis of FSI problems [24, 2].

There are some additional properties that solutions have. We have area conservation of the region $\Omega_{\rm i}$ which follows from the incompressibility condition (1.3) and condition (1.6). More concretely,

$$\frac{d}{dt}\left|\Omega_{\rm i}\right| = 0,\ \left|\Omega_{\rm i}\right| = \frac{1}{2}\int_{\mathbb{S}^1}\left(X\partial_\theta Y - Y\partial_\theta X\right)d\theta,\ \boldsymbol{X} = (X,Y)^{\rm T}. \tag{1.14}$$

Also, any solution of a Peskin problem with force given by $\boldsymbol{F}(\boldsymbol{X}) = \partial_\theta\left(T\left(\left|\partial_\theta \boldsymbol{X}\right|\right)\boldsymbol{\tau}(\boldsymbol{X})\right)$ has the following energy identity:

$$\frac{d\mathcal{E}}{dt} = -\mathcal{D},\ \mathcal{E} = \int_{\mathbb{S}^1} E(\left|\partial_\theta \boldsymbol{X}\right|)d\theta,\ \frac{dE}{ds} = \mathcal{T}(s),\ \mathcal{D} = \int_{\mathbb{R}^2}\left|\nabla \boldsymbol{u}\right|^2 d\boldsymbol{x}, \tag{1.15}$$

which may be most easily seen from (1.9). One can multiply the Stokes equation by $\boldsymbol{u}$, integrate by parts, and use (1.6). The integration by parts can be justified given sufficient regularity.

1.1 Problem Formulations under Different Force Densities

Up until now, we have assumed that our force density $\boldsymbol{F}(\boldsymbol{X})$ is generic and of the form (1.1). Our manuscript is split into related problems whose differences stem solely on the choice of tension T. We first consider the case where the tension is given by $T(s) = s$. In this case, the force density can be written as

$$\boldsymbol{F}(\boldsymbol{X}) = \partial_\theta^2 \boldsymbol{X}.$$

Using this, we can formally integrate by parts to rewrite equation (1.13) as

$$\partial_t \boldsymbol{X}(\theta, t) = -p.v. \int_{\mathbb{S}^1} \partial_{\theta'} G(\boldsymbol{X} - \boldsymbol{X}') \partial_{\theta'} \boldsymbol{X}' d\theta', \tag{1.16}$$

where this integral is to be understood in the principle value sense. We compute

$$\partial_{\theta'} G(\boldsymbol{X} - \boldsymbol{X}') = \frac{1}{4\pi} \frac{\Delta \boldsymbol{X} \cdot \partial_{\theta'} \boldsymbol{X}'}{|\Delta \boldsymbol{X}|^2} + \frac{1}{4\pi} \partial_{\theta'} \left(\frac{\Delta \boldsymbol{X} \otimes \Delta \boldsymbol{X}}{|\Delta \boldsymbol{X}|^2} \right), \tag{1.17}$$

where $\Delta \boldsymbol{X} = \boldsymbol{X} - \boldsymbol{X}'$. Note that to leading order,

$$\frac{\Delta \boldsymbol{X} \cdot \partial_{\theta'} \boldsymbol{X}'}{|\Delta \boldsymbol{X}|^2} \approx \frac{1}{\theta - \theta'},$$

when $|\theta - \theta'| \ll 1$. This motivates the idea that we may remove the Hilbert transform from the kernel. For a function f defined on $\mathbb{S}^1$, the Hilbert transform of f is defined as

$$(\mathcal{H} f)(\theta) = \frac{1}{2\pi} p.v. \int_{\mathbb{S}^1} \cot\left(\frac{\theta - \theta'}{2} \right) f(\theta') d\theta'.$$

Note that if $\boldsymbol{X}(\theta) = (r\cos\theta, r\sin\theta)^T$ is a uniformly parameterized circle of radius r, then

$$\begin{aligned} p.v. \int_{\mathbb{S}^1} \partial_{\theta'} \log |\Delta \boldsymbol{X}| f' d\theta' &= p.v. \int_{\mathbb{S}^1} \partial_{\theta'} \left(\log \left(4r \sin\left(\frac{\theta - \theta'}{2} \right) \right) \right) f' d\theta' \\ &= \int_{\mathbb{S}^1} \cot\left(\frac{\theta - \theta'}{2} \right) f' d\theta' = 2\pi \mathcal{H} f, \end{aligned}$$

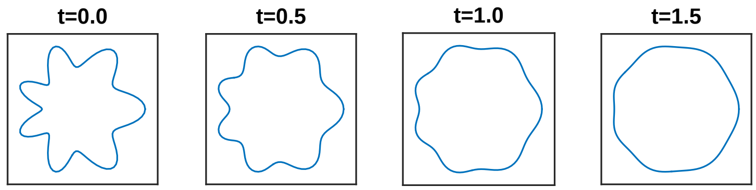

Figure 1.2: A sample evolution of an initial curve under the semilinear flow problem.

which supports the notion that the Hilbert transform may be the principle component of our kernel. We therefore rewrite our system as

$$\begin{aligned} &\partial_t \boldsymbol{X} = \Lambda \boldsymbol{X} + \mathcal{R}(\boldsymbol{X}),\ \Lambda \boldsymbol{X} = -\frac{1}{4}\mathcal{H}(\partial_\theta \boldsymbol{X}), \\ &\mathcal{R}(\boldsymbol{X}) = -\frac{1}{4\pi}\int_{\mathbb{S}^1}\left(\left(\frac{\Delta \boldsymbol{X}\cdot\partial_{\theta'}\boldsymbol{X}'}{|\Delta \boldsymbol{X}|^2} - \frac{1}{2}\cot\left(\frac{\theta-\theta'}{2}\right)\right) + \partial_{\theta'}\left(\frac{\Delta \boldsymbol{X}\otimes\Delta \boldsymbol{X}}{|\Delta \boldsymbol{X}|^2}\right)\right)\partial_{\theta'}\boldsymbol{X}'d\theta'. \end{aligned} \tag{1.18}$$

It is our hope that our system can be understood as a lower order perturbation away from principle linear evolution driven by Λ. This method of extracting the principal linear part in interfacial fluid problems is known as the *small scale decomposition* (Λ should control behavior at higher spatial wave number, or behavior at small spatial scales, and hence its name) and was introduced in [14] for the study of Hele-Shaw and water wave problems. In the context of numerical computation, the small scale decomposition allows for the removal of numerical stiffness; the stiff principal linear part is treated with an implicit numerical scheme while the remainder term is treated explicitly. Application of the small scale decomposition to IB problems can be found in [16, 15], although the small scale decomposition found in these papers seems to be slightly different from the one used in this manuscript, even taking into account the fact that they deal with the dynamic Stokes/Navier Stokes system. In Section 4.3.4, we use the small scale decomposition to develop a numerical scheme to computationally verify some of our theoretical results. The sample simulation in Figure 1.2 was generated using this algorithm.

Our approach to proving well-posedness is to turn (1.18) into an integral equation, a standard technique used in the study of parabolic equations [12, 21, 31]:

$$\boldsymbol{X}(t) = e^{\Lambda t}\boldsymbol{X}_0 + \int_0^t e^{\Lambda(t-s)}\mathcal{R}(\boldsymbol{X}(s))ds. \tag{1.19}$$

Here, $e^{\Lambda t}$ is the semigroup generated by Λ and $\boldsymbol{X}_0$ is the initial configuration. The success of this method depends upon whether or not Λ is the leading order operator. For this to be the case, it must equivalently be true that $\mathcal{R}(\boldsymbol{X})$ is, in some precise sense, a lower order term. This depends inherently on the choice of function spaces to work in. For this problem, we will construct solutions in the space of Hölder continuous functions, $C^{k,\gamma}$. Here k is an integer and $0 < \gamma < 1$. We will use $C^{k,\gamma}$ to refer both to functions in $\mathbb{R}$ as well as $\mathbb{R}^2$. We will show that $\mathcal{R}$ does belong to a smoother Hölder space. For this reason, we will refer to the problem stemming from this force density as the *semilinear Peskin problem*.

However, we cannot do this if we have a more generic force density $\boldsymbol{F}(\boldsymbol{X})$ with arbitrary tension T. We may still rewrite equation 1.13 by formally integrating by parts as

$$\begin{aligned}\partial_t \boldsymbol{X} = &-\frac{1}{4\pi} p.v. \int_{\mathbb{S}^1} \frac{\Delta \boldsymbol{X} \cdot \partial_{\theta'} \boldsymbol{X}'}{|\Delta \boldsymbol{X}|^2} \left(\mathcal{T}\left(|\partial_{\theta'} \boldsymbol{X}'|\right) \frac{\partial_{\theta'} \boldsymbol{X}'}{|\partial_{\theta'} \boldsymbol{X}'|}\right) d\theta' \\ &+ \frac{1}{4\pi} \int_{\mathbb{S}^1} \partial_{\theta'} \left(\frac{\Delta \boldsymbol{X} \otimes \Delta \boldsymbol{X}}{|\Delta \boldsymbol{X}|^2}\right) \left(\mathcal{T}\left(|\partial_{\theta'} \boldsymbol{X}'|\right) \frac{\partial_{\theta'} \boldsymbol{X}'}{|\partial_{\theta'} \boldsymbol{X}'|}\right) d\theta'.\end{aligned} \tag{1.20}$$

As before, we may wish to remove the Hilbert transform from the first kernel; nothing prevents us from adding and subtracting it from the kernel. In doing so, we have

$$\begin{aligned}\partial_t \boldsymbol{X} &= -\frac{1}{4\pi} \mathcal{H}\left(T\left(|\partial_\theta \boldsymbol{X}|\right) \frac{\partial_\theta \boldsymbol{X}}{|\partial_{\theta'} \boldsymbol{X}|}\right) + \mathcal{R}(\boldsymbol{X}(\theta)) \\ \mathcal{R}(\boldsymbol{X}) &= -\frac{1}{4\pi} \int_{\mathbb{S}^1} \left(\frac{\Delta \boldsymbol{X} \cdot \partial_\theta \boldsymbol{X}'}{|\Delta \boldsymbol{X}|^2} - \frac{1}{2}\cot\left(\frac{\theta - \theta'}{2}\right)\right) \left(T\left(|\partial_{\theta'} \boldsymbol{X}'|\right) \frac{\partial_{\theta'} \boldsymbol{X}'}{|\partial_{\theta'} \boldsymbol{X}'|}\right) d\theta' \\ &\quad - \frac{1}{4\pi} \int_{\mathbb{S}^1} \partial_{\theta'} \left(\frac{\Delta \boldsymbol{X} \otimes \Delta \boldsymbol{X}}{|\Delta \boldsymbol{X}|^2}\right) \left(T\left(|\partial_{\theta'} \boldsymbol{X}'|\right) \frac{\partial_{\theta'} \boldsymbol{X}'}{|\partial_{\theta'} \boldsymbol{X}'|}\right) d\theta'.\end{aligned}$$

However, this first term

$$-\frac{1}{4\pi} \mathcal{H}\left(\mathcal{T}\left(|\partial_\theta \boldsymbol{X}|\right) \frac{\partial_\theta \boldsymbol{X}}{|\partial_\theta \boldsymbol{X}|}\right)$$

is not linear in $\boldsymbol{X}$, and so we cannot use the same semilinear theory, but must appeal to nonlinear theory instead. We will instead rewrite our problem (1.20) as

$$\partial_t \boldsymbol{X} = F(\boldsymbol{X}), \;\; \boldsymbol{X}(0) = \boldsymbol{X}_0,$$

for F given by

$$\begin{aligned} F(\boldsymbol{X}) = &-\frac{1}{4\pi}\int_{\mathbb{S}^1} \frac{\Delta\boldsymbol{X}\cdot\partial_{\theta'}\boldsymbol{X}'}{|\Delta\boldsymbol{X}|^2}\left(\mathcal{T}\left(\left|\partial_{\theta'}\boldsymbol{X}'\right|\right)\frac{\partial_{\theta'}\boldsymbol{X}'}{|\partial_{\theta'}\boldsymbol{X}'|}\right)d\theta' \\ &+\frac{1}{4\pi}\int_{\mathbb{S}^1}\partial_{\theta'}\left(\frac{\Delta\boldsymbol{X}\otimes\Delta\boldsymbol{X}}{|\Delta\boldsymbol{X}|^2}\right)\left(\mathcal{T}\left(\left|\partial_{\theta'}\boldsymbol{X}'\right|\right)\frac{\partial_{\theta'}\boldsymbol{X}'}{|\partial_{\theta'}\boldsymbol{X}'|}\right)d\theta', \end{aligned}$$

and then, consider the Gâteaux derivative of F at the initial data $\boldsymbol{X}_0$ applied to $\boldsymbol{X}$,

$$A = \partial_{\boldsymbol{X}}F(\boldsymbol{X}_0)\boldsymbol{X} := \frac{d}{d\epsilon}F(\boldsymbol{X}_0+\epsilon\boldsymbol{X})\big|_{\epsilon=0}. \tag{1.21}$$

Using this, we may rewrite our system as

$$\partial_t \boldsymbol{X} = A\boldsymbol{X} + G(\boldsymbol{X}), \;\; \boldsymbol{X}(0) = \boldsymbol{X}_0,$$

where $G(\boldsymbol{X}) = F(\boldsymbol{X}) - A\boldsymbol{X}$. Then, as in equation (1.19), we write our solution as

$$\boldsymbol{X}(t) = e^{At}\boldsymbol{X}_0 + \int_0^t e^{A(t-s)}G(\boldsymbol{X}(s))ds. \tag{1.22}$$

The hope here is that for small times $\boldsymbol{X}(t) \approx \boldsymbol{X}_0$ and $A\boldsymbol{X} \approx F(\boldsymbol{X})$, so that while G has the same mapping properties as A, it can be thought of as "small". This technique is standard in the study of nonlinear parabolic problems, see [21]. Thus, when considering generic force densities of this type, we will refer to the problem as the *fully nonlinear Peskin problem*. We must first establish that solutions to our problem can even be written as (1.22). This requires us to prove rigorously that operator A generates an analytic semigroup on appropriate spaces. The choice of function space must be made very carefully. In fact, most choices of function space will not even close equation (1.22) in the same space, rendering it impossible to find a fixed point within the space. We will work within the space of little Hölder continuous functions $h^{k,\gamma}(\mathbb{S}^1)$, the completion of smoother functions within $C^{k,\gamma}(\mathbb{S}^1)$, and build our solution in $C([0,T]; h^{1,\gamma}(\mathbb{S}^1))$.

1.2 Discussion of Results

In this section, we provide an overview of the major results within this manuscript. We first state and discuss the results obtained for the semilinear Peskin problem which make up the bulk of this document. Afterwards, we present a statement of the fully nonlinear result.

1.2.1 Discussion of Semilinear Results

Before we can state the definition of a solution to the semilinear Peskin problem, we must introduce the following quantity defined for a function $\boldsymbol{X}(\theta) \in C^1(\mathbb{S}^1)$ [22]:

$$|\boldsymbol{X}|_* = \inf_{\theta,\theta' \in \mathbb{S}^1, \theta \neq \theta'} \frac{|\boldsymbol{X}(\theta) - \boldsymbol{X}(\theta')|}{|\theta - \theta'|}. \tag{1.23}$$

Note that $|\boldsymbol{X}|_* = 0$ if and only if $|\partial_\theta \boldsymbol{X}| = 0$ at some point or if the curve self-intersects, i.e. $\boldsymbol{X}(\theta) = \boldsymbol{X}(\theta')$ for some $\theta \neq \theta'$. Thus, $\boldsymbol{X}$ defines a non-degenerate simple closed curve if and only if $|\boldsymbol{X}|_* > 0$. Let $C^n([0,T]; C^{k,\gamma}(\mathbb{S}^1))$ be the space of C^n functions of t, $0 \leq t \leq T$ with values in $C^{k,\gamma}(\mathbb{S}^1)$. We define two notions of solutions to the semilinear Peskin problem.

Definition 1.2.1 (Mild Solution)**.** *Let* $\boldsymbol{X}(t) \in C([0,T]; C^{1,\gamma}(\mathbb{S}^1)), 0 < \gamma < 1$ *and* $|\boldsymbol{X}(t)|_* > 0$ *for* $0 \leq t \leq T$*. Then,* $\boldsymbol{X}$ *is a* mild solution *to the semilinear Peskin problem with initial value* $\boldsymbol{X}(0) = \boldsymbol{X}_0$ *if it satisfies equation* (1.19) *for* $0 < t \leq T$ *and* $\boldsymbol{X}(t) \to \boldsymbol{X}_0$ *in* $C^{1,\gamma}(\mathbb{S}^1)$ *as* $t \to 0$*.*

Definition 1.2.2 (Strong Solution)**.** *Let* $\boldsymbol{X}(t) \in C([0,T]; C^{1,\gamma}(\mathbb{S}^1)) \cap C^1([0,T]; C^{0,\gamma}(\mathbb{S}^1))$, $0 < \gamma < 1$ *and* $|\boldsymbol{X}(t)|_* > 0$ *for* $0 \leq t \leq T$*. Then,* $\boldsymbol{X}$ *is a* strong solution *to the semilinear Peskin problem with initial value* $\boldsymbol{X}(0) = \boldsymbol{X}_0$ *if it satisfies equation* (1.16) *for* $0 < t \leq T$ *and* $\boldsymbol{X}(t) \to \boldsymbol{X}_0$ *in* $C^{1,\gamma}(\mathbb{S}^1)$ *as* $t \to 0$*.*

We now state our result on the local well-posedness of the semilinear Peskin problem.

Theorem 1.2.3. *Consider the semilinear Peskin problem with initial value* $\boldsymbol{X}_0 \in h^{1,\gamma}(\mathbb{S}^1)$, $0 < \gamma < 1$ *with* $|\boldsymbol{X}_0|_* > 0$*. Then, we have the following.*

(i) For some time $T > 0$ *depending on* $\boldsymbol{X}_0$*, there is a mild solution* $\boldsymbol{X}(t) \in C([0,T]; C^{1,\gamma}(\mathbb{S}^1))$*.*

(ii) Suppose $\boldsymbol{X}(t) \in C([0,T]; C^{1,\gamma}(\mathbb{S}^1))$ *is a mild solution to the semilinear Peskin problem. Then, this solution is unique within the class* $C([0,T]; C^{1,\gamma}(\mathbb{S}^1))$*.*

(iii) Let $\boldsymbol{X}(t) \in C([0,T]; C^{1,\gamma}(\mathbb{S}^1))$ be the mild solution to the Peskin problem with initial data $\boldsymbol{X}_0$. Then, there is an $\epsilon > 0$ such that, for all initial data $\boldsymbol{Y}_0$ satisfying $\|\boldsymbol{X}_0 - \boldsymbol{Y}_0\|_{C^{1,\gamma}} \leq \epsilon$, there is a mild solution $\boldsymbol{X}(t; \boldsymbol{Y}_0) \in C([0,T]; C^{1,\gamma}(\mathbb{S}^1))$. Furthermore, $\boldsymbol{X}(t; \boldsymbol{Y}_0)$ is a continuous function of $\boldsymbol{Y}_0 \in C^{1,\gamma}(\mathbb{S}^1)$ with values in $C([0,T]; C^{1,\gamma}(\mathbb{S}^1))$.

(iv) The function $\boldsymbol{X}(t)$ is a mild solution on $[0,T]$ if and only if it is a strong solution on $[0,T]$.

We prove the existence of a mild solution (1.19) by a contraction mapping argument. There are two ingredients to the proof of Theorem 1.2.3. The first ingredient is a set of estimates in Hölder norms of the semigroup operator generated by Λ in (1.18). The semigroup $e^{t\Lambda}$ satisfies estimates typical of linear parabolic semigroups such as the heat propagator, except that $\Lambda = -\frac{1}{4}\mathcal{H}\partial_\theta$ has the effect of taking only one spatial derivative as the Hilbert transform is a bounded operator on the space of Hölder continuous functions. This is in contrast to the Laplacian which takes two spatial derivatives. These estimates are found by an explicit representation of $e^{t\Lambda}$ as a convolution operator with the Poisson kernel, as discussed in Section 3.1.1. These estimates can be obtained using abstract semigroup theory. However, for the semilinear problem, working directly with explicit expression for the semigroup yields stronger, more transparent results.

The second ingredient is a class of smoothing estimates on the nonlinear remainder $\mathcal{R}(\boldsymbol{X}(s))$; we show that $\mathcal{R}(\boldsymbol{X}) : C^{1,\gamma} \mapsto C^{2\gamma} = C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}$ for $\gamma \in (0,1) \backslash \frac{1}{2}$. This shows, in essence, that $\mathcal{R}$ has the effect of taking $1 + \gamma - 2\gamma = 1 - \gamma$ derivatives. As discussed earlier, Λ behaves like taking one derivative, and $\mathcal{R}$ is thus genuinely lower order by γ derivatives. This allows us to view Λ as the principal part of the evolution, making it possible to use Duhamel's formula (1.19). These crucial smoothing estimates on the remainder $\mathcal{R}$ arise from the structure of the kernel: (a) the components of the kernel are composed of rational functions of finite differences of $\boldsymbol{X}$ and its derivatives and (b) the kernel is a perfect derivative in θ'. Our bounds on the components of the kernel, found in Section 2.2, rely on careful, albeit elementary, estimates on these rational functions. Finally, since the kernel is a perfect derivative, it allows us to gain an extra γ in our Hölder estimate, which is used to close the argument. We remark here that our local existence theory is close to optimal, in the sense that $\mathcal{R}$ takes only γ fewer derivatives than Λ, and $\gamma > 0$ can be made arbitrarily small. We are thus at the edge of applicability of semilinear parabolic techniques; any meaningful improvement on our local existence theory may require fundamentally different techniques.

Once we have proven the existence of the mild solution, we show that our mild solution has the expected $C^1([0,T];C^{0,\gamma}(\mathbb{S}^1))$ regularity. Since the solution satisfies the differential form of the equation pointwise, we are able to conclude the existence of a unique strong solution.

Our next result shows that the mild solution and its time derivative are arbitrarily smooth for any positive time.

Theorem 1.2.4. *Consider the mild (strong) solution $\boldsymbol{X}$ of Theorem 1.2.3. The function $\boldsymbol{X}$ is in $C^1([\epsilon,T];C^n(\mathbb{S}^1))$ for any $n\in\mathbb{N}$ and $\epsilon>0$.*

The proof of Theorem 1.2.4 is found in Section 4.2. Since the remainder $\mathcal{R}(\boldsymbol{X})$ is a nonlinear smoothing kernel acting on $\partial_\theta \boldsymbol{X}$, in order to prove higher regularity, we introduce a class of integral kernels that allow us to move derivatives in θ on the nonlinear kernel into derivatives in θ' acting on $\partial_{\theta'}\boldsymbol{X}'$. Since the error from this operation is lower order, the regularity improvement from the semigroup lets us gain arbitrarily high regularity in space. The corresponding smoothness in time arises from equation (1.18). Higher regularity in time should be achievable using similar techniques, but we do not pursue it in this paper.

An immediate corollary of this result is that the strong solution constructed in Theorem 1.2.3 is classical in the sense that it satisfies the jump, IB and BI formulations of the equations pointwise. The precise definitions and these solutions are discussed in Section 4.2.2.

Given the regularity of solutions to the semilinear Peskin problem, we may now justify energy identity (1.15) which, for our choice of tension, is as follows:

$$\frac{d\mathcal{E}}{dt}=-\mathcal{D},\ \mathcal{E}=\frac{1}{2}\int_{\mathbb{S}^1}|\partial_\theta\boldsymbol{X}|^2\,d\theta,\ \mathcal{D}=\int_{\mathbb{R}^2}|\nabla\boldsymbol{u}|^2\,d\boldsymbol{x}, \tag{1.24}$$

Corollary 1.2.5. *The notions of classical jump, IB, BI solutions and mild (strong) solution are equivalent. Furthermore, the mild (strong) solution satisfies area conservation* (1.14) *and energy identity* (1.24).

Any classical solution, which by definition should possess sufficient smoothness, is clearly a strong solution. This then proves the unique existence of classical solutions and the equivalence of the three formulations of the semilinear Peskin problem.

In Section 4.3 we study the equilibria of the Peskin problem and their stability. The computation of the equilibria is performed using the jump formulation of the equations, which is made possible by Corollary 1.2.5. There are four symmetries about solutions to the semilinear Peskin problem. It is well known that any translation or rotation of a solution is again a solution. It is also easy to see from (1.16) that there is also a dilation invariance:

$$\text{If } \boldsymbol{X}(\theta,t) \text{ is a solution, so is } a\boldsymbol{X}(\theta,t) \text{ for any } a>0. \tag{1.25}$$

We find that the only equilibria are circles in which the material points are evenly spaced:

$$\begin{aligned} &\boldsymbol{X}(\theta) = A\boldsymbol{e}_{\mathrm{r}} + B\boldsymbol{e}_{\mathrm{t}} + C_1\boldsymbol{e}_x + C_2\boldsymbol{e}_y,\ A^2+B^2>0,\\ &\boldsymbol{e}_{\mathrm{r}} = \begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix},\ \boldsymbol{e}_{\mathrm{t}} = \begin{pmatrix}-\sin(\theta)\\ \cos(\theta)\end{pmatrix},\ \boldsymbol{e}_x = \begin{pmatrix}1\\ 0\end{pmatrix},\ \boldsymbol{e}_y = \begin{pmatrix}0\\ 1\end{pmatrix}. \end{aligned} \tag{1.26}$$

For later reference, we let $\widehat{\mathcal{V}}$ denote the above set of circular equilibria and let $\mathcal{V}$ be the linear space in $C^{1,\gamma}(\mathbb{S}^1)$ spanned by the above 4 basis vectors $\boldsymbol{e}_{\mathrm{r,t},x,y}$.

We then investigate the stability of these steady states. We first study the linearization of the evolution operator at the above uniformly parametrized circles. By dilation, translation and rotation invariance discussed above, the linearized operator $\mathcal{L}$ is the same at every circle. This makes our analysis considerably simpler than it would be otherwise and also leads to stronger results. In Section 4.3.2, we explicitly compute the spectrum of $\mathcal{L}$ and obtain the decay properties of the semigroup $e^{t\mathcal{L}}$. The operator $\mathcal{L}$ has a four-dimensional kernel that coincides with $\mathcal{V}$. Except for the 0 eigenvalue corresponding to the kernel $\mathcal{V}$, all eigenvalues are negative and real, and the leading non-zero principal eigenvalue is $-1/4$. In fact, $\mathcal{L}$ is a self-adjoint operator on $L^2(\mathbb{S}^1;\mathbb{R}^2)$, the space of square-integrable functions with values in $\mathbb{R}^2$. For two functions $\boldsymbol{v},\boldsymbol{w}\in L^2(\mathbb{S}^1;\mathbb{R}^2)$, we define the standard L^2 inner product as:

$$\langle \boldsymbol{v},\boldsymbol{w}\rangle = \int_{\mathbb{S}^1} \boldsymbol{v}(\theta)\cdot\boldsymbol{w}(\theta)d\theta. \tag{1.27}$$

In Section 4.3.3, we establish nonlinear stability of the circular equilibria. To state our result we introduce some notation. Let $\mathcal{P}$ be the L^2 projection on to $\mathcal{V}$ and Π its complementary projection:

$$\mathcal{P}\boldsymbol{X} = \frac{1}{2\pi}\sum_{\ell=\mathrm{r,t},x,y}\langle \boldsymbol{X},\boldsymbol{e}_\ell\rangle\boldsymbol{e}_\ell,\ \Pi\boldsymbol{X} = \boldsymbol{X}-\mathcal{P}\boldsymbol{X}. \tag{1.28}$$

The above L^2 projections are clearly well-defined operators on Hölder spaces as well. Notice that $\mathcal{P}\boldsymbol{X}\in\mathcal{V}$ is a circle so long as it does not degenerate to a point. Thus, the magnitude of $\Pi\boldsymbol{X}$ measures the distance from the set of circular equilibria. We let the norm on $\mathcal{V}$, which we denote by $\|\cdot\|_{\mathcal{V}}$, to be the standard Euclidean $\mathbb{R}^4$ norm with respect to the coordinate vectors $\boldsymbol{e}_{\mathrm{r,t},x,y}$. We have the following result.

Theorem 1.2.6. *Circles with evenly spaced material points as given in* (1.26) *are the only equilibria of the Peskin system. Furthermore, there is a constant* $\rho_0 > 0$ *that depends only on* γ *with the following properties. Consider a mild solution* $\boldsymbol{X}(t)$ *to the Peskin problem with initial data* $\boldsymbol{X}_0 \in h^{1,\gamma}(\mathbb{S}^1)$*. Let* $R > 0$ *be the radius of* $\mathcal{P}\boldsymbol{X}_0$*, and suppose* $\|\Pi\boldsymbol{X}_0\|_{C^{1,\gamma}} \leq \rho_0 R$*. Then, the solution to the Peskin problem is defined for all positive time and converges to a circle* $\boldsymbol{Z}_\infty \in \widehat{\mathcal{V}}$*. Furthermore, we have the following estimates.*

(i) For $t \geq 0$*, we have:*

$$\|\Pi\boldsymbol{X}(t)\|_{C^{1,\gamma}} \leq C\,\|\Pi\boldsymbol{X}_0\|_{C^{1,\gamma}}\, e^{-t/4}, \tag{1.29}$$

$$\|\mathcal{P}\boldsymbol{X}(t) - \boldsymbol{Z}_\infty\|_{\mathcal{V}} \leq \frac{C}{R}\,\|\Pi\boldsymbol{X}_0\|_{C^{1,\gamma}}^2\, e^{-t/2}, \tag{1.30}$$

where the above constants C *depend only on* γ*. As an immediate consequence of the above results, we have:*

$$\|\boldsymbol{X}(t) - \boldsymbol{Z}_\infty\|_{C^{1,\gamma}} \leq C\,\|\Pi\boldsymbol{X}_0\|_{C^{1,\gamma}}\, e^{-t/4}, \tag{1.31}$$

where C *depends only on* γ*.*

(ii) For any $n \in \mathbb{N}, n \geq 2$ *and* $t \geq \epsilon > 0$*, we have:*

$$\|\Pi\boldsymbol{X}(t)\|_{C^n} \leq C\,\|\Pi\boldsymbol{X}_0\|_{C^{1,\gamma}}\, e^{-t/4}, \tag{1.32}$$

where the constant C *depends only on* n, ϵ *and* γ*. An immediate consequence of this and* (1.30) *is that, for any* $n \in \mathbb{N}, n \geq 2$ *and* $t \geq \epsilon > 0$*,*

$$\|\boldsymbol{X}(t) - \boldsymbol{Z}_\infty\|_{C^n} \leq C\,\|\Pi\boldsymbol{X}_0\|_{C^{1,\gamma}}\, e^{-t/4} \tag{1.33}$$

where the above constant C *depends only on* n, ϵ *and* γ*.*

To prove this theorem, we first obtain a Lipschitz estimate on the derivative of the nonlinear remainder term. We then use a standard Lyapunov-Perron type fixed point argument on time-exponentially weighted spaces to obtain the exponential decay to circular equilibria. Note here that, in all of the above estimates, the right and left hand side of the inequalities scale proportionally with dilation, as they should given dilation invariance of the semilinear Peskin system.

In many results of this type, it is only possible to prove that the decay rate can be made arbitrarily close but not equal to the value of the real part of the leading non-zero eigenvalue (in our case, $-1/4$) [21, 31]. Here, an explicit calculation of the kernel $e^{t\mathcal{L}}$ allows us to obtain a sharp linear decay estimate, which in turn leads to this sharp result. Inequality (1.30) indicates that the projected dynamics on the set of equilibria given by $\mathcal{P}\boldsymbol{X}(t)$ is exponentially approaching the limiting circle $\boldsymbol{Z}_\infty$ at twice the rate of $-1/2 = 2 \times (-1/4)$. This somewhat unexpected result is a consequence of the fact that the zero-eigenspace $\mathcal{V}$ and the set of equilibria $\widehat{\mathcal{V}}$ essentially coincide, which in turn is a reflection of the four-dimensional group of symmetries acting on the Peskin system. Finally, exponential decay in higher norms given in (1.32) follows by combining (1.29) with the parabolic regularity estimates of Section 4.2.

In Section 4.3.4, we computationally verify the exponential decay estimates stated in Theorem 1.2.6. The numerical scheme we develop is a boundary integral method based on the small scale decomposition in (1.18) and is second order accurate in time and spectrally accurate in space. We see that the exponential decay rate of $\Pi\boldsymbol{X}(t)$ and $\mathcal{P}\boldsymbol{X}(t)$ is indeed asymptotically $-1/4$ and $-1/2$, respectively.

Finally, in Section 4.4, we address issues of global behavior. It is convenient to define the notion of a solution on half-open time intervals. Let the space $C^n([0,T');C^{k,\gamma}(\mathbb{S}^1))$ to be the union of all $C^n([0,T];C^{k,\gamma}(\mathbb{S}^1))$ with $0 < T < T'$. Here, $T' > 0$ is allowed to be finite or $T' = \infty$.

Definition 1.2.7 (Solution on half-open time intervals)**.** *If $\boldsymbol{X}(t) \in C([0,T');C^{1,\gamma}(\mathbb{S}^1))$, $0 < \gamma < 1$ and $|\boldsymbol{X}(t)|_* > 0$ for $0 \le t < T'$, $\boldsymbol{X}(t)$ is a mild solution if the restriction of $\boldsymbol{X}(t)$ to any interval $[0,T],\ 0 < T < T'$ is a mild solution.*

Given initial data $\boldsymbol{X}_0 \in h^{1,\gamma}(\mathbb{S}^1)$, define the maximal interval of existence of a mild solution $\tau_{\max}(\boldsymbol{X}_0)$ as follows. Let $\mathcal{S}(\boldsymbol{X}_0)$ be the set of all $T > 0$ such that there exists a mild solution $\boldsymbol{X}(t) \in C([0,T];C^{1,\gamma}(\mathbb{S}^1))$. We let:

$$\tau_{\max}(\boldsymbol{X}_0) = \sup_{T \in \mathcal{S}(\boldsymbol{X}_0)} T.$$

Note that, if $\boldsymbol{X}_1(t) \in C([0,T_1];C^{1,\gamma}(\mathbb{S}^1))$ and $\boldsymbol{X}_2(t) \in C([0,T_2];C^{1,\gamma}(\mathbb{S}^1))$ for $0 < T_1 < T_2$ with the same initial data $\boldsymbol{X}_0$, $\boldsymbol{X}_1(t) = \boldsymbol{X}_2(t)$ up to $t = T_1$ by the uniqueness result in Theorem 1.2.3. Thus, one may speak of the unique mild solution $\boldsymbol{X}(t)$ with initial data $\boldsymbol{X}_0 \in h^{1,\gamma}(\mathbb{S}^1)$ defined up to any $t < \tau_{\max}(\boldsymbol{X}_0)$. Therefore, $\boldsymbol{X}(t) \in C([0,\tau_{\max}(\boldsymbol{X}_0));C^{1,\gamma}(\mathbb{S}^1))$. It is important to note here that $\boldsymbol{X}(t)$ cannot be in

$C([0,\tau_{\max}(\boldsymbol{X}_0)];C^{1,\gamma}(\mathbb{S}^1))$. If so, we will be able to extend the solution further by Theorem 1.2.3, contradicting the definition of $\tau_{\max}$. If $\tau_{\max}(\boldsymbol{X}_0)=\infty$, we say that the solution is global.

To state our results, we introduce the γ-*deformation ratio*:

$$\varrho_\gamma(\boldsymbol{X}):=\frac{\|\partial_\theta \boldsymbol{X}\|_{C^{0,\gamma}}}{|\boldsymbol{X}|_*}.$$

This quantity is invariant under translation, rotation and dilation. Note that

$$\varrho_\gamma(\boldsymbol{X})=\frac{\|\partial_\theta \boldsymbol{X}\|_{C^{0,\gamma}}}{|\boldsymbol{X}|_*}\geq\frac{\sup_{\theta\in\mathbb{S}^1}|\partial_\theta \boldsymbol{X}|}{\inf_{\theta\in\mathbb{S}^1}|\partial_\theta \boldsymbol{X}|}\geq 1.$$

The γ-deformation ratio is thus always greater than 1, and we may replace the last inequality with an equality if $\boldsymbol{X}$ is a uniformly parametrized circle. In this sense, the γ-deformation ratio measures the degree to which $\boldsymbol{X}$ is deformed from a uniform circle configuration.

Theorem 1.2.8. *Given initial data $\boldsymbol{X}_0\in h^{1,\gamma}(\mathbb{S}^1)$, consider the mild solution $\boldsymbol{X}(t)\in C([0,\tau_{\max}(\boldsymbol{X}_0));C^{1,\gamma}(\mathbb{S}^1))$.*

(i) Suppose $\tau_{\max}(\boldsymbol{X}_0)<\infty$. Then,

$$\lim_{t\to\tau_{\max}(\boldsymbol{X}_0)}\varrho_\alpha(\boldsymbol{X}(t))=\infty,$$

for any $0<\alpha<1$. In particular, the maximal existence time $\tau_{\max}(\boldsymbol{X}_0)$ does not depend on γ (the space $C^{1,\gamma}$ in which the mild solution is considered).

(ii) Suppose the solution is global, that is $\tau_{\max}(\boldsymbol{X}_0)=\infty$. Suppose furthermore that

$$\sup_{t\geq 0}\varrho_\alpha(\boldsymbol{X}(t))<\infty$$

for some $0<\alpha\leq\gamma$. Then, the solution $\boldsymbol{X}(t)$ converges exponentially to a uniformly parametrized circle as described in Theorem 1.2.6.

In the proof of this theorem, the energy and area conservation identities (1.24) and (1.14) play a key role. The deformation ratio together with area conservation gives a lower bound on $|\boldsymbol{X}|_*$ whereas the deformation ratio bound and energy decay give an upper bound on the norm $\|\boldsymbol{X}\|_{C^{1,\alpha}}$.

Item (i) above is a consequence of these bounds on $|\boldsymbol{X}|_*$ and $\|\boldsymbol{X}\|_{C^{1,\alpha}}$ as well as the regularity results of Section 4.2. An interesting point about item (i) is that *all* deformation ratios

$\varrho_\alpha(\boldsymbol{X}), 0 < \alpha < 1$ must tend to ∞ as t reaches the maximal existence time. In particular, this shows that the maximal existence time is independent of the value of γ in $C^{1,\gamma}(\mathbb{S}^1)$, the space in which we consider the mild solution. This leads us to conjecture that the 0-deformation ratio, $\varrho_0(\boldsymbol{X}) := \frac{\|\partial_\theta \boldsymbol{X}\|_{C^0}}{|\boldsymbol{X}|_*}$, would blow up at the finite extinction time.

Item (ii) states that a global solution with bounded deformation ratio converges to a circle. If the deformation ratio is bounded, $|\boldsymbol{X}|_*$ and $\|\boldsymbol{X}\|_{C^{1,\alpha}}$ are bounded by energy decay and area conservation as discussed above. This shows that the orbit $\boldsymbol{X}(t)$ is relatively compact in any space $C^{1,\beta}(\mathbb{S}^1), 0 < \beta < \alpha$, meaning that $\boldsymbol{X}(t)$ has a well-defined ω-limit set in $C^{1,\beta}(\mathbb{S}^1)$. Viewing the energy as a Lyapunov function, one can then conclude that the ω-limit set must consist only of stationary circles. This, together with Theorem 1.2.6, allows us to establish the desired result.

1.2.2 Discussion of the Fully Nonlinear Result

This concludes the results for the semilinear Peskin problem and so we move onto the result for the fully nonlinear Peskin problem. The results above comprise a very thorough examination of the semilinear Peskin problem and detail many aspects of solutions. We start the process of recreating these results or analogues thereof. Because of the inexplicit form of the tension, we must change methods slightly to embrace abstract results instead of calculating bounds by hand as needed. By application of Theorem 8.3.4 from [21], we prove the following result.

Theorem 1.2.9. *Suppose the tension $T(s) \in h^{1,\gamma}(\mathbb{R})$ is such that both $T(s) > 0$ and $T'(s) > 0$. Consider the fully nonlinear Peskin problem* (1.20) *with initial data $\boldsymbol{X}_0 \in h^{1,\gamma}(\mathbb{S}^1)$ with $|\boldsymbol{X}_0|_* > 0$. Then, we have the following.*

(i) There exists some $\tau > 0$ such that (1.20) *has a unique solution $\boldsymbol{X}(t) \in C([0,\tau]; h^{1,\gamma}(\mathbb{S}^1)) \cap C^1([0,\tau], h^{\gamma}(\mathbb{S}^1))$.*

(ii) There exists some $\epsilon > 0$ such that if $\boldsymbol{Y}_0 \in h^{1,\gamma}(\mathbb{S}^1)$ with $\|\boldsymbol{X}_0 - \boldsymbol{Y}_0\|_{h^{1,\gamma}} < \epsilon$, then (1.20) *has a unique solution $\boldsymbol{X}(t; \boldsymbol{Y}_0) \in C([0,\tau]; h^{1,\gamma}(\mathbb{S}^1)) \cap C^1([0,\tau], h^{\gamma}(\mathbb{S}^1))$, where $\tau > 0$ is as in statement (i), corresponding to initial data $\boldsymbol{Y}_0$.*

This theorem parallels Theorem 1.2.3 from the semilinear case with the solution now built in the space of little Hölder continuous functions. The first statement guarantees the solution is both a mild solution and a strong solution to the fully nonlinear problem and the second statement gives continuity with respect to initial data.

In order to investigate the existence of solutions by use of Duhamel's equation as in (1.22), we need to first verify that the Gâteaux derivative of the right hand side of (1.20) generates an analytic semigroup on "suitable spaces". There is no explicit expression for the semigroup in this nonlinear setting and so we use the abstract theory laid our in [21]. In Section 3.2, using perturbative methods, we are able to leverage the semigroup generated in the semilinear problem to show that our linearized operator also generates an analytic semigroup on both $C^{n,\gamma}$ and $h^{n,\gamma}$.

The method of proving Theorem 1.2.9 is not terribly different from the semilinear case - Theorem 8.3.4 in [21] also constructs a solution by building a contraction mapping of the operator defined by Duhamel's equation. As was stated earlier, the difficulty in the fully nonlinear case is that only special choices of function spaces will actually close the equation back in the original space. Such spaces are referred to as optimal regularity spaces (see [21] chapter 8 for an in-depth discussion on the matter). The little Hölder continuous functions have two properties which make them ideal for this situiation. First, they form a set of interpolation spaces between each other - the big Hölder continuous functions also enjoy this property. However, the little Hölder continuous functions are also the completion of smoother function in the Hölder continuous functions. Thus, for specific choices $D(A)$, the domain of operator, we will have $h^\alpha \subset \overline{D(A)}$ for some α. This second property allows us to achieve the initial data as $t \to 0$. We will prove in proposition 3.2.7 that the space of little Hölder continuous functions coincides with a set of certain optimal regularity spaces for appropriate choices of $D(A)$.

Note that the solution to the fully nonlinear Peskin problem is both a mild and a strong solution. However, as stated, we are still $1-\gamma$ derivatives short to proving that the solution satisfies the boundary integral form (1.13). We will require further regularity results on the solution of the nonlinear Peskin problem before we can determine whether or not it satisfies the other formulations.

Higher regularity in time is almost immediate given lemmas 2.2.5, 2.2.6 and 2.2.8. Regularity in space will require a new argument however, as the linearized operator A is not a perfect convolution and maps $A : h^{n,\gamma} \mapsto h^{0,\gamma}$ for any $n > 1$. Thus, A does not carry the regularity of the function it is being applied to given its explicit dependence on the initial data $\boldsymbol{X}_0 \in h^{1,\gamma}$. Therefore, we cannot use the same method that we used in the semilinear problem.

1.3 Related Results

A preprint [20] considers the Peskin problem and establishes local well-posedness of strong solutions with initial data $\boldsymbol{X}_0 \in H^{5/2}(\mathbb{S}^1)$ and $|\boldsymbol{X}_0|_* > 0$ which generates a unique solution in $C([0,T]; H^{5/2}(\mathbb{S}^1)) \cap H^1((0,T); H^2(\mathbb{S}^1))$ for some $T > 0$. Local existence follows by energy arguments, use of Fourier multiplier methods, and an application of the Schauder fixed point theorem. The authors also show that a solution with initial data close to a circular equilibrium converges in the $H^{5/2}(\mathbb{S}^1)$ norm to a circular equilibrium at some exponential rate. This is established with the help of the energy identity. We note that Theorem 1.2.3 is a full derivative weaker. Furthermore, it is unclear if the strong solutions in [20] strictly satisfy the energy equality or satisfy the classical form of (1.18).

The Peskin problems considered here are simple cases of a much wider class of immersed boundary problems in which a thin elastic structure interacts with the surrounding fluid. While we have started the process of working with more generic constitutive laws for elasticity, there are, of course, other options that can be considered. We could also modify the problem to the case where the fluids filling the interior and exterior are different by differing their viscosities. Doing this would introduce a second integral equation for the force density. We may also replace the Stokes equation with the Navier-Stokes equation for the fluid equations in the interior and exterior domains, possibly with different viscosities and mass densities. We may also consider 3D problems in which the elastic force is generated by a 2D membrane. All of these generalizations are important in applications, and it would therefore be more descriptive to refer to our problem as the 2D Peskin-Stokes problem.

We note that the choice $E(s) = s, \mathcal{T}(s) = 1$ leads to the 2D surface tension problem. In this case, $\mathcal{E}$ is simply the length of the elastic filament, and surface tension acts to decrease the interfacial length. The energy law

$$\frac{d\mathcal{E}}{dt} = -\mathcal{D},\ \mathcal{E} = \int_{\mathbb{S}^1} |\partial_\theta \boldsymbol{X}|\, d\theta,\ \frac{dE}{ds} = \mathcal{T}(s) = 1,\ \mathcal{D} = \int_{\mathbb{R}^2} |\nabla \boldsymbol{u}|^2\, d\boldsymbol{x},$$

makes it clear that the Peskin problem and the surface tension problem are different. Surface tension only depends on the curvature of Γ, and therefore only on the shape (or geometry) of Γ. In contrast, in the Peskin problem, the force depends on the material parametrization. In particular, stretching the interface leads to a force in the Peskin problem but not in the surface tension problem, and in this sense, the interface in the surface tension problem is *not* elastic.

For example, any θ parametrization of the circle will be an equilibrium configuration for the surface tension problem, but only the uniform Lagrangian parametrization of the circle is an equilibrium configuration for the Peskin problem.

The surface tension problem itself has many variants. The analytical study of the one-phase problem, in which the fluid equations (Stokes or Navier Stokes) are satisfied in the interior region Ω_{i} only, was initiated by Solonnikov [36], and has since been taken up by many authors. The two-phase surface tension problem, in which the exterior region Ω_{e} is also filled with a Stokes of Navier-Stokes fluid, possibly of different viscosity and mass density, has also been studied by many authors, though the results are somewhat more recent. We refer the reader to [29, 30, 34] where an extensive list of references on these problems can be found. We also point to several recent results on problem with structures with more complicated energies interacting with the surrounding fluid [4, 5, 25].

There are other problems in fluid mechanics which bear similarities to ours; the closest of which is the Muskat problem. In the simplest setup in two dimensions, the Muskat problem features two fluids in porous media whose dynamics are governed by Darcy's law. For nearly flat interfaces, the linearization of the Muskat problem has the same symbol as the semilinear Peskin problem considered here, and one expects similar local well-posedness and stability results so long as condition $|\boldsymbol{X}|_* > 0$ holds, see for example [35, 1, 10, 7, 6, 3].

Chapter 2

Calculus Estimates

The evolution of our system is dictated by

$$\begin{aligned}\boldsymbol{X}_t = &-\frac{1}{4\pi}\int_{\mathbb{S}^1} \log|\Delta\boldsymbol{X}|\,\partial_{\theta'}\boldsymbol{F}(\partial_{\theta'}\boldsymbol{X}')d\theta' \\ &+\frac{1}{4\pi}\int_{\mathbb{S}^1}\left(\frac{\Delta\boldsymbol{X}\otimes\Delta\boldsymbol{X}}{|\Delta\boldsymbol{X}|^2}\right)\partial_{\theta'}\boldsymbol{F}(\partial_{\theta'}\boldsymbol{X}')d\theta',\end{aligned}$$

for some function $\boldsymbol{F}(\partial_\theta\boldsymbol{X})$ as given in (1.1). Before we break into different settings determined by the elasticity law, we first do a thorough study of the kernels of these integral operators.

The kernels of these diffeo-integral operators,

$$-\partial_{\theta'}\log|\Delta\boldsymbol{X}| = \frac{\Delta\boldsymbol{X}\cdot\partial_{\theta'}\boldsymbol{X}'}{|\Delta\boldsymbol{X}|^2} \tag{2.1}$$

and

$$\partial_{\theta'}\left(\frac{\Delta\boldsymbol{X}\otimes\Delta\boldsymbol{X}}{|\Delta\boldsymbol{X}|^2}\right) \tag{2.2}$$

belong to or map into a more general class of functions. We will introduce this class of functions and prove specific properties and bounds that members of this class enjoy. But first, some notation.

2.1 Notation

We first introduce some standard function spaces. Let $C^k(\mathbb{S}^1), k = 0, 1, 2, \ldots$ be the space of functions on $\mathbb{S}^1$ with k continuous derivatives. Define the norms on these spaces in the usual way:

$$\|u\|_{C^k} = \sum_{j=0}^{k} [u]_{C^j} \ , \ [u]_{C^k} = \sup_{\theta \in \mathbb{S}^1} \left| \partial_\theta^k u \right|.$$

A function $u \in C^0(\mathbb{S}^1)$ is in the Hölder space $C^{0,\gamma}(\mathbb{S}^1), \ 0 < \gamma < 1$ if

$$\sup_{\theta,\theta' \in \mathbb{S}^1} \frac{|u(\theta) - u(\theta')|}{|\theta - \theta'|^\gamma} < \infty.$$

Given the continuity of u, we may also restrict the range of θ and θ' to $|\theta - \theta'| < 1$, for instance. Define the $C^{0,\gamma}$ norm as:

$$\|u\|_{C^{0,\gamma}} = \|u\|_{C^0} + [u]_{C^{0,\gamma}} \ , \ [u]_{C^{0,\gamma}} = \sup_{|\theta-\theta'|<1} \frac{|u(\theta) - u(\theta')|}{|\theta - \theta'|^\gamma}.$$

Now, consider the space of Little Hölder continuous functions $h^{0,\gamma}$. A function $u \in h^{0,\gamma}(\mathbb{S}^1)$ if $u \in C^{0,\gamma}(\mathbb{S}^1)$ and

$$\lim_{\theta \to \theta'} \frac{|u(\theta) - u(\theta')|}{|\theta - \theta'|^\gamma} = 0.$$

The space of Little Hölder Continuous functions is the closure of any smoother space C^α, $\alpha > \gamma$ in $C^{0,\gamma}$; this includes any C^k spaces. The norm on $h^{0,\gamma}$ is the same as the norm on $C^{0,\gamma}$.

A function $u \in C^k(\mathbb{S}^1), k = 0, 1, 2, \ldots$ is in $C^{k,\gamma}(\mathbb{S}^1), 0 < \gamma < 1$ if the k-th derivative of u is in $C^{0,\gamma}(\mathbb{S}^1)$ and we define the norm on this space by:

$$\|u\|_{C^{k,\gamma}} = \|u\|_{C^k} + \left[\partial_\theta^k u\right]_{C^{0,\cdot}}$$

For any function $F(\theta)$ on the circle $\mathbb{S}^1$, we define

$$\Delta F = F(\theta) - F(\theta').$$

We also let $\partial_\theta F$ be the derivative of F evaluated at θ where as $\partial_{\theta'} F'$ will be the derivative evaluated at θ'. We will use the same notation for vector-valued functions on the circle. We will be considering the difference quotient of functions evaluated at θ and $\theta + \eta$. Without loss of generality, assume that $0 \leq \theta - \eta/2 < \theta < \theta + \eta < \theta + 3\eta/2 \leq 2\pi$. This can be achieved since all of our functions are periodic. We will often split $\mathbb{S}^1$ into two parts,

$$\begin{aligned} \mathcal{I}_s &:= (\theta - \eta/2, \theta + 3\eta/2) \\ \mathcal{I}_f &:= \mathbb{S}^1 \backslash \mathcal{I}_s \end{aligned} \tag{2.3}$$

In the following, we drop the dependence on θ in the definition of $\mathcal{I}_s$.

For a function $f(\theta, \theta')$, we use the notation:

$$\begin{aligned}(\mathcal{T}_\eta f)(\theta, \theta') &= f(\theta + \eta, \theta'),\\ (\triangle_\eta f)(\theta, \theta') &= (\mathcal{T}_\eta f - f)(\theta, \theta') = f(\theta + \eta, \theta') - f(\theta, \theta').\end{aligned} \tag{2.4}$$

A product rule follows from these definitions,

$$\triangle_\eta (f(\theta, \theta') g(\theta, \theta')) = (\triangle_\eta f)(\theta, \theta')(\mathcal{T}_\eta g)(\theta, \theta') + f(\theta, \theta')(\triangle_\eta g)(\theta, \theta'). \tag{2.5}$$

Also, define the space $C^k(I; X)$ to be the space C^k functions on the possibly unbounded interval I, taking values in Banach space X.

Hölder continuous functions have some special properties which make them extremely useful to us in this work. In particular, they are interpolation spaces of each other. Both the little Hölder continuous functions and the Hölder continuous functions have this property, but they are different types of interpolation spaces. We first introduce the two types of interpolation spaces we will be looking at.

For a set of Banach spaces $Y \subset X$, we define the operator $K(t, x, X, Y)$ as

$$K(t, x, X, Y) = \inf_{x = a+b, a \in X, b \in Y} \|a\|_X + t \|b\|_y .$$

Using this function, we define the interpolation space $(X, Y)_{\sigma,\infty}$ as

$$(X, Y)_{\sigma,\infty} = \{x \in X : t \mapsto t^{-\sigma} K(t, x, X, Y) \in L^\infty(0, 1)\} \tag{2.6}$$

with norm

$$\|x\|_{(X,Y)_{\sigma,\infty}} = \left\|t^{-\sigma} K(t, x, X, Y)\right\|_{L^\infty(0,\infty)} .$$

Further, we define the space $(X, Y)_\sigma$ as

$$(X, Y)_\sigma = \{x \in X : \lim_{t \to 0} t^{-\sigma} K(t, x, X, Y) = 0\}. \tag{2.7}$$

This space as the same norm as $(X, Y)_{\sigma,\infty}$ and is clearly closed in $(X, Y)_{\sigma,\infty}$.

For any $0 < \alpha < \beta < \gamma$ with $\alpha, \beta, \gamma \notin \mathbb{N}$ the inclusion $C^\gamma \subset C^\beta \subset C^\alpha$ clearly holds. It is in fact the case that

$$C^\beta \simeq (C^\alpha, C^\gamma)_{\sigma,\infty}, \;\; \sigma = \frac{\beta - \alpha}{\gamma - \alpha}. \tag{2.8}$$

Further,

$$h^\beta \simeq (h^\alpha, h^\gamma)_{\sigma}, \;\; \sigma = \frac{\beta - \alpha}{\gamma - \alpha}. \tag{2.9}$$

Here, we must be very careful that the exponents α, β, γ do not fall on integer values. We can lift this restriction by looking at the Hölder-Zygmund spaces instead of Hölder spaces. However, this will not be necessary for us.

Frequently, we will need to make sense of generic operators A. Let us define the domain of an operator A as $D(A)$.

2.2 Homogeneous Quotients of Hölder Continuous Functions

In what follows, we prove a sequence of lemmas on quotients of differences of $C(I; C^{1,\gamma})$ functions. We then generalize these lemmas $C^k(I; C^{k,\gamma})$ functions and provide some mapping properties of different derivative operators. Consider the following class of functions:

Definition 2.2.1. *A function $f(t, \theta, \theta')$ is said to belong to class $\mathcal{S}^H_{k,n,\gamma}$ on interval I if it is of the form*

$$g(t, \theta, \theta') = \prod_{i=0}^{n} \left(\frac{\Delta Z_i}{|\boldsymbol{Z}_0|} \right)^{\alpha_i} \left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|} \right)^{\beta_i},$$

for some set of functions $\{\boldsymbol{Z}_i\}_{i=0}^n$, with $\boldsymbol{Z}_i(t,\theta) = (Z_i(t,\theta), W_i(t,\theta)) \in C^k(I; C^{n,\gamma})$, and $|\boldsymbol{Z}_0(t)|_* > 0$ for all $t \in I$, and some α_i, $\beta_i \in \mathbb{N} \cup \{0\}$.

We will frequently suppress the variable t in what follows and will occasionally write $\mathcal{S}^H := \mathcal{S}^H_{0,1,\gamma}$. We will build up properties on $\mathcal{S}^H$ by first studying the prototypical example of a function in this class.

Lemma 2.2.2. *Suppose the functions $\boldsymbol{Z}(\theta) = (Z(\theta), W(\theta))$ and $V(\theta)$ belong to $C^{1,\gamma}(\mathbb{S}^1)$ with $|\boldsymbol{Z}|_* > 0$. Let*

$$\phi(\theta, \theta') = \frac{\Delta V}{|\Delta \boldsymbol{Z}|}.$$

(i) The following estimates hold for ϕ and its derivatives.

$$|\phi| \leq \frac{\|V\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*}, \tag{2.10}$$

$$|\partial_\theta \phi|, |\partial_{\theta'} \phi| \leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*^2} \left|\theta - \theta'\right|^{\gamma - 1}, \tag{2.11}$$

$$|\partial_{\theta'} \partial_\theta \phi| \leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^3} \left|\theta - \theta'\right|^{\gamma - 2}. \tag{2.12}$$

If, in addition, $V = Z$ or $V = W$, we have the following estimates:

$$|\phi| \leq 1, \tag{2.13}$$

$$|\partial_\theta \phi|, |\partial_{\theta'} \phi| \leq C \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*} \left|\theta - \theta'\right|^{\gamma - 1}, \tag{2.14}$$

$$|\partial_{\theta'} \partial_\theta \phi| \leq C \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2} \left|\theta - \theta'\right|^{\gamma - 2}. \tag{2.15}$$

(ii) Suppose $0 < h < |\theta - \theta' + h/2|$ and $0 < \theta + h < 2\pi$. Then, the following estimates hold.

$$|\triangle_h(\partial_\theta \phi)| \leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^3} \left(h^\gamma \left|\theta - \theta'\right|^{-1} + h \left|\theta - \theta'\right|^{\gamma - 2}\right). \tag{2.16}$$

$$\left|\triangle_h(\partial_\theta' \partial_\theta \phi)\right| \leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^3} \left(h^\gamma \left|\theta - \theta'\right|^{-2} + h \left|\theta - \theta'\right|^{\gamma - 3}\right). \tag{2.17}$$

If in addition, $V = Z$ or $V = W$, we have:

$$|\triangle_h(\partial_\theta \phi)| \leq C \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2} \left(h^\gamma \left|\theta - \theta'\right|^{-1} + h \left|\theta - \theta'\right|^{\gamma - 2}\right). \tag{2.18}$$

$$\left|\triangle_h(\partial_\theta' \partial_\theta \phi)\right| \leq C \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2} \left(h^\gamma \left|\theta - \theta'\right|^{-2} + h \left|\theta - \theta'\right|^{\gamma - 3}\right). \tag{2.19}$$

In the above, the positive constant C do not depend on $V, \boldsymbol{Z}, \theta, \theta'$ or h.

Proof. Let us first prepare some elementary estimates. First, we have:

$$|\Delta V| \leq [V]_{C^1} \left|\theta - \theta'\right| \leq \|V\|_{C^{1,\gamma}} \left|\theta - \theta'\right|. \tag{2.20}$$

A similar estimate holds for $\boldsymbol{Z}$. We have, by definition of $|\boldsymbol{Z}|_*$,

$$|\Delta \boldsymbol{Z}| \geq |\boldsymbol{Z}|_* \left|\theta - \theta'\right|. \tag{2.21}$$

We also have:

$$\begin{aligned}\left|\frac{\Delta V}{\theta - \theta'} - \partial_\theta V\right| &= \left|\frac{1}{\theta - \theta'}\int_0^1 \frac{d}{ds} V(s\theta + (1-s)\theta')ds - \partial_\theta V\right| \\ &\leq \int_0^1 \left|\partial_\theta V(\theta' + s(\theta - \theta')) - \partial_\theta V(\theta)\right| ds \\ &\leq [V]_{C^{1,\gamma}} \int_0^1 |1-s|^\gamma \left|\theta - \theta'\right|^\gamma ds \leq \|V\|_{C^{1,\gamma}} \left|\theta - \theta'\right|^\gamma .\end{aligned} \tag{2.22}$$

A similar bound holds when $\partial_\theta V$ is replaced by $\partial_{\theta'} V'$ or for V replaced by $\boldsymbol{Z}$, all with the same proof.

We first consider the first three bounds in item (i). The bound (2.10) follows from (2.20) and (2.21). For (2.11), we prove the bound for $\partial_\theta \phi$. The bound for $\partial_{\theta'} \phi$ can be obtained in exactly the same way. After some calculation, we obtain:

$$\begin{aligned}\partial_\theta \phi &= \frac{1}{|\Delta \boldsymbol{Z}|^3}\Big(\partial_\theta V \,|\Delta \boldsymbol{Z}|^2 - \Delta V \Delta \boldsymbol{Z} \cdot \partial_\theta \boldsymbol{Z}\Big) = A + B, \\ A &= \frac{1}{|\Delta \boldsymbol{Z}|}\left(\partial_\theta V - \frac{\Delta V}{\theta - \theta'}\right),\ B = \frac{1}{|\Delta \boldsymbol{Z}|^3}\left(\Delta V \Delta \boldsymbol{Z} \cdot \left(\frac{\Delta \boldsymbol{Z}}{\theta - \theta'} - \partial_\theta \boldsymbol{Z}\right)\right).\end{aligned} \tag{2.23}$$

Using (2.22) and (2.21), we obtain:

$$|A| \leq \frac{\|V\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*} \left|\theta - \theta'\right|^{\gamma - 1}. \tag{2.24}$$

We also have:

$$|B| \leq \frac{|\Delta V|}{|\Delta \boldsymbol{Z}|^2}\left|\frac{\Delta \boldsymbol{Z}}{\theta - \theta'} - \partial_\theta \boldsymbol{Z}\right| \leq \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*^2} \left|\theta - \theta'\right|^{\gamma - 1}, \tag{2.25}$$

where we used the Cauchy-Schwarz inequality in the first inequality and (2.20), (2.21) and (2.22) (as applied to $\boldsymbol{Z}$) in the last inequality. Noting that $\|\boldsymbol{Z}\|_{C^{1,\gamma}} \geq |\boldsymbol{Z}|_*$, we may combine (2.24) and (2.25) to obtain (2.11).

Let us now prove bound (2.12). We have, after some calculation:

$$\begin{aligned}
&\partial_{\theta'}\partial_\theta\phi = D + E + F,\\
&\quad D = \frac{1}{|\Delta\boldsymbol{Z}|^5}\left(\partial_\theta V \Delta\boldsymbol{Z}\cdot\partial_{\theta'}\boldsymbol{Z}'\,|\Delta\boldsymbol{Z}|^2 - \Delta V(\Delta\boldsymbol{Z}\cdot\partial_\theta\boldsymbol{Z})(\Delta\boldsymbol{Z}\cdot\partial_{\theta'}\boldsymbol{Z}')\right),\\
&\quad E = \frac{1}{|\Delta\boldsymbol{Z}|^5}\left(\partial_{\theta'} V' \Delta\boldsymbol{Z}\cdot\partial_{\theta}\boldsymbol{Z}\,|\Delta\boldsymbol{Z}|^2 - \Delta V(\Delta\boldsymbol{Z}\cdot\partial_\theta\boldsymbol{Z})(\Delta\boldsymbol{Z}\cdot\partial_{\theta'}\boldsymbol{Z}')\right),\\
&\quad F = \frac{1}{|\Delta\boldsymbol{Z}|^5}\left(\Delta V \partial_\theta\boldsymbol{Z}\cdot\partial_{\theta'}\boldsymbol{Z}'\,|\Delta\boldsymbol{Z}|^2 - \Delta V(\Delta\boldsymbol{Z}\cdot\partial_\theta\boldsymbol{Z})(\Delta\boldsymbol{Z}\cdot\partial_{\theta'}\boldsymbol{Z}')\right).
\end{aligned} \tag{2.26}$$

We estimate D. We have:

$$\begin{aligned}
&D = D_1 + D_2,\\
&D_1 = \frac{1}{|\Delta\boldsymbol{Z}|^3}\left(\partial_\theta V - \frac{\Delta V}{\theta-\theta'}\right)\Delta\boldsymbol{Z}\cdot\partial_{\theta'}\boldsymbol{Z}',\\
&D_2 = \frac{1}{|\Delta\boldsymbol{Z}|^5}\Delta V(\Delta\boldsymbol{Z}\cdot\partial_{\theta'}\boldsymbol{Z}')\left(\Delta\boldsymbol{Z}\cdot\left(\frac{\Delta\boldsymbol{Z}}{\theta-\theta'} - \partial_\theta\boldsymbol{Z}\right)\right).
\end{aligned} \tag{2.27}$$

Using estimates (2.20), (2.21) and (2.22), we have:

$$|D_1| \le \frac{1}{|\Delta\boldsymbol{Z}|^2}\left|\partial_\theta V - \frac{\Delta V}{\theta-\theta'}\right|\left|\partial_{\theta'}\boldsymbol{Z}'\right| \le \frac{\|V\|_{C^{1,\gamma}}\|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*^2}\left|\theta-\theta'\right|^{\gamma-2}. \tag{2.28}$$

Likewise, for D_2, we have:

$$|D_2| \le \frac{\|V\|_{C^{1,\gamma}}\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^3}\left|\theta-\theta'\right|^{\gamma-2}.$$

Combining the above estimates and noting that $\|\boldsymbol{Z}\|_{C^{1,\gamma}} \ge |\boldsymbol{Z}|_*$, we have:

$$|D| \le C\frac{\|V\|_{C^{1,\gamma}}\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^3}\left|\theta-\theta'\right|^{\gamma-2}$$

In a similar fashion, E and F can be shown to satisfy the same bound. This concludes the proof of (2.12).

To obtain the alternate estimates in item (i) when $V = Z$ or $V = W$ note that:

$$|\Delta V| \le |\Delta\boldsymbol{Z}| \text{ if } V = Z \text{ or } V = W.$$

Bound (2.13) is a direct consequence of this. We may use this to improve the bound on B in (2.25) to obtain (2.14). We may also show that D_2 satisfies the bound (2.28) and hence obtain a better bound for D and similarly for E and F of (2.26). This yields (2.15).

Finally, we turn to item (ii). Let us first consider (2.16). Given expression (2.23), we may estimate $\triangle_h A$ and $\triangle_h B$ seperately. We have:

$$\begin{aligned}&\triangle_h A = A_1 + A_2,\\ &\quad A_1 = \frac{1}{|\Delta \boldsymbol{Z}|}\triangle_h\left(\partial_\theta V - \frac{\Delta V}{\theta-\theta'}\right),\ A_2 = \triangle_h\left(\frac{1}{|\Delta \boldsymbol{Z}|}\right)\mathcal{T}_h\left(\partial_\theta V - \frac{\Delta V}{\theta-\theta'}\right).\end{aligned}$$

We first estimate A_1.

$$|\triangle_h \partial_\theta V| \leq \|V\|_{C^{1,\gamma}} h^\gamma.$$

Furthermore, similarly to the calculation in (2.22), we have:

$$\begin{aligned}\left|\triangle_h\left(\frac{\Delta V}{\theta-\theta'}\right)\right| &\leq \int_0^1 \left|\partial_\theta V((1-s)(\theta+h)+s\theta') - \partial_\theta V((1-s)\theta+s\theta')\right| ds\\ &\leq [V]_{C^{1,\gamma}}\int_0^1 (1-s)^\gamma h^\gamma ds \leq \|V\|_{C^{1,\gamma}} h^\gamma.\end{aligned}$$

Using the above two relations and (2.21), we thus have:

$$|A_1| \leq C\frac{\|V\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*} h^\gamma \left|\theta-\theta'\right|^{-1} \tag{2.29}$$

We now estimate A_2.

$$\triangle_h\left(\frac{1}{|\boldsymbol{Z}|}\right) = h\int_0^1 \partial_\theta\left(\frac{1}{|\boldsymbol{Z}|}\right)((\theta-\theta')+sh)ds$$

Since

$$\left|\partial_\theta\left(\frac{1}{|\boldsymbol{Z}|}\right)\right| = \frac{|\Delta \boldsymbol{Z}\cdot\partial_\theta \boldsymbol{Z}|}{|\Delta \boldsymbol{Z}|^3} \leq \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*^2}\left|\theta-\theta'\right|^{-2}$$

we have:

$$\triangle_h\left(\frac{1}{|\boldsymbol{Z}|}\right) \leq \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*^2}\int_0^1 \left|\theta+sh-\theta'\right|^{-2} ds.$$

Note that:

$$\frac{|\theta-\theta'|}{|\theta-\theta'+sh|} \leq 1 + \frac{sh}{|\theta-\theta'+sh|} \leq 1 + \frac{sh}{|\theta-\theta'+h/2|-h/2} \leq 1+2s \leq 3. \tag{2.30}$$

Thus,

$$\triangle_h\left(\frac{1}{|\boldsymbol{Z}|}\right) \leq C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*^2} h\left|\theta-\theta'\right|^{-2}.$$

Using the above and (2.22), we have:

$$|A_2| \leq C\frac{\|V\|_{C^{1,\gamma}}\|\boldsymbol{Z}\|_{C^{1,\gamma}} h}{|\boldsymbol{Z}|_*^2}\left|\theta-\theta'\right|^{-2}\left|\theta+h-\theta'\right|^\gamma.$$

In much the same way as in (2.30),

$$\frac{|\theta - \theta' + h|}{|\theta - \theta'|} \leq 3. \tag{2.31}$$

Thus,

$$|A_2| \leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*^2} h \left|\theta - \theta'\right|^{\gamma - 2}. \tag{2.32}$$

Combining (2.29) and (2.32) and using $|\boldsymbol{Z}|_* \leq \|\boldsymbol{Z}\|_{C^{1,\gamma}}$ we see that

$$|\triangle_h A| \leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*^2} \left(h^\gamma \left|\theta - \theta'\right|^{-2} + h \left|\theta - \theta'\right|^{\gamma-2}\right). \tag{2.33}$$

We turn to $\triangle_h B$. We have:

$$\triangle_h B = B_1 + B_2,$$
$$B_1 = \frac{\Delta V \Delta \boldsymbol{Z}}{|\Delta \boldsymbol{Z}|^3} \cdot \triangle_h \left(\partial_\theta \boldsymbol{Z} - \frac{\Delta \boldsymbol{Z}}{\theta - \theta'}\right),\ B_2 = \triangle_h \left(\frac{\Delta V \Delta \boldsymbol{Z}}{|\Delta \boldsymbol{Z}|^3}\right) \cdot \mathcal{T}_h \left(\partial_\theta \boldsymbol{Z} - \frac{\Delta \boldsymbol{Z}}{\theta - \theta'}\right).$$

In much the same way we obtained the estimates for A_1, for B_1, we have:

$$|B_1| \leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*^2} h^\gamma \left|\theta - \theta'\right|^{-1}. \tag{2.34}$$

We turn to B_2. We have:

$$\left|\partial_\theta \left(\frac{\Delta V \Delta \boldsymbol{Z}}{|\Delta \boldsymbol{Z}|^3}\right)\right| \leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^3} \left|\theta - \theta'\right|^{-2}, \tag{2.35}$$

where we used $|\boldsymbol{Z}|_* \leq \|\boldsymbol{Z}\|_{C^{1,\gamma}}$. Using the same procedure as was used forA_2, we have:

$$\left|\triangle_h \left(\frac{\Delta V \Delta \boldsymbol{Z}}{|\Delta \boldsymbol{Z}|^3}\right)\right| \leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^3} h \left|\theta - \theta'\right|^{-2}.$$

Combining this with (2.22) (as applied to $\boldsymbol{Z}$), we obtain:

$$|B_2| \leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^3} h \left|\theta - \theta'\right|^{\gamma-2}. \tag{2.36}$$

Combining (2.34) and (2.36), we obtain the estimate:

$$|\triangle_h B| \leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^3} \left(h^\gamma \left|\theta - \theta'\right|^{-2} + h \left|\theta - \theta'\right|^{\gamma-2}\right). \tag{2.37}$$

Combining (2.33) and (2.37), we obtain (2.16). When $V = Z$ or $V = W$, we can obtain the following bound in place of (2.35):

$$\left|\partial_\theta \left(\frac{\Delta V \Delta \boldsymbol{Z}}{|\Delta \boldsymbol{Z}|^3}\right)\right| \le C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*^2}\left|\theta-\theta'\right|^{-2}.$$

This allows us to prove (2.18).

Finally, we turn to (2.17) and (2.19). By (2.26), we may estimate the differences of D, E and F separately. To estimate $\triangle_h D$, we estimate $\triangle_h D_1$ and $\triangle_h D_2$ where $D_{1,2}$ are given in (2.27). The difference $\triangle_h D_1$ can be estimated in a similar way to $\triangle_h A$ above and $\triangle_h D_2$ similarly to $\triangle_h B$ above. The differences $\triangle_h E$ and $\triangle_h F$ can be estimated similarly to $\triangle_h D$. We omit the details. □

This lemma can be generalized to any function in the class $\mathcal{S}^H$

Lemma 2.2.3. *Suppose $g(\theta, \theta')$ is of class $\mathcal{S}^H$. Let*

$$N = \sum_{i=1}^n (\alpha_i + \beta_i),\ N_0 = \sum_{i=0}^n (\alpha_i + \beta_i) = (\alpha_0 + \beta_0) + N.$$

We have the following estimates.

$$\left|g(\theta,\theta')\right| \le \frac{C}{|\boldsymbol{Z}_0|_*^N}\prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i}, \tag{2.38}$$

$$\left|\partial_\theta g(\theta,\theta')\right|, \left|\partial_{\theta'} g(\theta,\theta')\right| \le C\frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+1}}\prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i}|\theta-\theta'|^{\gamma-1}, \tag{2.39}$$

$$\left|\partial_\theta\partial_{\theta'} g(\theta,\theta')\right| \le C\frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}_0|_*^{N+2}}\prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i}\left|\theta-\theta'\right|^{\gamma-2} \tag{2.40}$$

for some constant C which depends only on γ, the coefficients of different terms in g and N_0.

Proof. Given the assumption on the indices α_i, β_i and N, we have:

$$g = \prod_{i=0}^n \phi_i^{\alpha_i}\psi_i^{\beta_i},\ \phi_i = \frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|},\ \psi_i = \frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|}. \tag{2.41}$$

The functions ϕ_i and ψ_i satisfy the assumptions of Lemma 2.2.2. Thus, using (2.10) and (2.11), for $\phi_i, i \ge 1$, we have:

$$|\phi_i| \le \frac{\|\boldsymbol{Z}_i\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*},\quad |\partial_\theta \phi_i| \le C\frac{\|\boldsymbol{Z}_i\|_{C^{1,\gamma}}\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^2}\left|\theta-\theta'\right|^{\gamma-1}, \tag{2.42}$$

where we used the fact that $\|\boldsymbol{Z}_i\|_{C^{1,\gamma}}$ dominates the norm of its components. The function $\psi_i, i \geq 1$ satisfies exactly the same bounds. For ϕ_0, from (2.13) and (2.14) we have the bounds:

$$|\phi_0| \leq 1, \quad |\partial_\theta \phi_i| \leq C\frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*}\left|\theta - \theta'\right|^{\gamma-1}. \tag{2.43}$$

The same bound holds for ψ_0. Inequality (2.38) is immediate from (2.41), (2.42) and (2.43). We now prove (2.39). Taking the derivative of g with respect to θ, we obtain:

$$\begin{aligned} \partial_\theta g &= \sum_{k=0}^{n}(\alpha_k A_k + \beta_k B_k), \\ A_k &= (\partial_\theta \phi_k)\phi_k^{\alpha_k - 1}\psi_k^{\beta_k}\prod_{i\neq k}\phi_i^{\alpha_i}\psi_i^{\beta_i}, \quad B_k = (\partial_\theta \psi_k)\phi_k^{\alpha_k}\psi_k^{\beta_k-1}\prod_{i\neq k}\phi_i^{\alpha_i}\psi_i^{\beta_i}. \end{aligned} \tag{2.44}$$

Let us bound $A_k, k \geq 1$.

$$|A_k| \leq |\partial_\theta \phi_k|\,|\phi_k|^{\alpha_k-1}\,|\psi_k|^{\beta_k}\prod_{i\neq k}|\phi_i|^{\alpha_i}\,|\psi_i|^{\beta_i} \leq C\frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+1}}\prod_{i=i}^{n}\|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i}\,|\theta-\theta'|^{\gamma-1},$$

where we used (2.42) and (2.43)(and the same bounds for the ψ_i's). The terms $B_k, k \geq 0$ as well as A_0 satisfy exactly the same inequality. Inequality (2.39) is now immediate. The inequality for $\partial_{\theta'} g$ can be obtained in exactly the same way.

We now prove (2.40). Using the notation of (2.44), we have:

$$\partial_{\theta'}\partial_\theta g = \sum_{k=0}^{N}\left(\alpha_k \partial_{\theta'} A_k + \beta_k \partial_{\theta'} B_k\right) \tag{2.45}$$

Let us estimate $\partial_{\theta'} A_k$ when $k \geq 1$ (and k for which $\alpha_k \neq 0$). We see that:

$$\partial_{\theta'} A_k = (\partial_{\theta'}\partial_\theta \phi_k) g_k + \partial_\theta \phi_k \partial_{\theta'} g_k, \quad g_k = \phi_k^{\alpha_k-1}\psi_k^{\beta_k}\prod_{i\neq k}\phi_i^{\alpha_i}\psi_i^{\beta_i}. \tag{2.46}$$

Recall from Lemma 2.2.2 that

$$|\partial_{\theta'}\partial_\theta \phi_k| \leq C\frac{\|\boldsymbol{Z}_k\|_{C^{1,\gamma}}\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}_0|_*^3}\left|\theta-\theta'\right|^{\gamma-2}.$$

Thus, combining (2.42) and the above, we have:

$$|(\partial_{\theta'}\partial_\theta \phi_k) g_k| \leq C\frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}_0|_*^{N+2}}\prod_{i=1}^{n}\|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i}\,|\theta-\theta'|^{\gamma-2}$$

Note that g_k satisfies the hypothesis of the present lemma for g with α_k replaced by $\alpha_k - 1$. We may thus use (2.39) directly to estimate $\partial g_k/\partial\theta'$. Thus, using this together with (2.42),

$$|\partial_\theta \phi_k \partial_{\theta'} g_k| \leq C \frac{\|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}} |\theta - \theta'|^{2\gamma-2}$$

Note that

$$|\theta - \theta'|^{2\gamma-2} \leq C|\theta - \theta'|^{\gamma-2}$$

for some constant C depending only on γ. Combining the above two estimates, we obtain a bound on $\partial A_k/\partial\theta'$. Noting that A_0 and $B_k, k \geq 0$ satisfy the same estimates, we obtain the desired result. $\square$

Lemma 2.2.4. *Let $g(\theta, \theta')$ as in Lemma 2.2.3, $0 < \eta < |\theta - \theta' + \eta/2|$ and $0 < \theta + \eta < 2\pi$.*

$$\left|\triangle_\eta \partial_{\theta'} g(\theta, \theta')\right| \leq C \frac{\|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}} \eta \left|\theta - \theta'\right|^{\gamma-2} \tag{2.47}$$

$$\left|\triangle_\eta \partial_\theta \partial_{\theta'} g(\theta, \theta')\right| \leq C \frac{\|\boldsymbol{Z}_0\|^3_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+3}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}} \left(\eta^\gamma \left|\theta - \theta'\right|^{-2} + \eta \left|\theta - \theta'\right|^{\gamma-3}\right), \tag{2.48}$$

where the constant C depends only on γ, the coefficients of terms of g and N_0.

Proof. Let us first prove (2.47). We have:

$$\begin{aligned}
|\triangle_h \partial_{\theta'} g| &= h \int_0^1 \left|\partial_{\theta'} \partial_\theta g(\theta + sh, \theta')\right| ds \\
&\leq C \frac{\|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}} Ch \int_0^1 \left(\left|\theta + sh - \theta'\right|^{\gamma-2} + \left|\theta + sh - \theta'\right|^{2\gamma-2}\right) ds \\
&\leq C \frac{\|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}} Ch \left(\left|\theta - \theta'\right|^{\gamma-2} + \left|\theta - \theta'\right|^{2\gamma-2}\right).
\end{aligned}$$

We next prove (2.48). Recall that $\partial_\theta \partial_{\theta'} g$ can be written as (2.45). We use the notation there. Consider bounding $\triangle_h \partial_{\theta'} A_k$. From (2.46), we have:

$$\begin{aligned}
\triangle_h \partial'_\theta A_k &= A_{k1} + A_{k2} + A_{k3} + A_{k4}, \\
A_{k1} &= (\partial_{\theta'} \partial_\theta \phi_k) \triangle_h g_k, \ A_{k2} = \triangle_h \left(\partial_{\theta'} \partial_\theta \phi_k\right) \mathcal{T}_h g_k, \\
A_{k3} &= (\partial_\theta \phi_k) \triangle_h \partial_{\theta'} g_k, \ A_{k4} = \triangle_h (\partial_\theta \phi_k) \mathcal{T}_h \partial_{\theta'} g_k.
\end{aligned}$$

Suppose $k \geq 1$. Let us estimate A_{k1} (for k such that $\alpha_k \neq 0$). Note that

$$\begin{aligned}
|\triangle_h g_k| &\leq h \int_0^1 \left|\partial_\theta g_k(\theta + sh, \theta')\right| ds \\
&\leq C \frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^N} \prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\widetilde{\alpha}_i+\beta_i} \, h \int_0^1 \left|\theta + sh - \theta'\right|^{\gamma-1} ds \\
&\leq C \frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^N} \prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\widetilde{\alpha}_i+\beta_i} \, h \left|\theta - \theta'\right|^{\gamma-1},
\end{aligned} \tag{2.49}$$

where $\widetilde{\alpha}_i = \alpha_i$ if $i \neq k$ and $\widetilde{\alpha}_k = \alpha_k - 1$ otherwise. Combining this with (2.40), we have:

$$|A_{k1}| \leq C \frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^3}{|\boldsymbol{Z}_0|_*^{N+3}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} \, h \left(\left|\theta - \theta'\right|^{2\gamma-3} + \left|\theta - \theta'\right|^{3\gamma-3}\right).$$

Turn next to A_{k2}. Using (2.17) and (2.38), we have:

$$|A_{k2}| \leq |\triangle_h \partial_{\theta'} \partial_\theta \phi_k| \, |\mathcal{T}_h g_k| \leq C \frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^{N+2}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} \left(h^\gamma \left|\theta - \theta'\right|^{-2} + h \left|\theta - \theta'\right|^{\gamma-3}\right)$$

For A_{k3}, using (2.11) and (2.49) we have:

$$\begin{aligned}
|A_{k3}| &\leq |\partial_{\theta'} \phi_k| \, |\triangle_h \partial_{\theta'} g_k| \\
&\leq C \frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^3}{|\boldsymbol{Z}_0|_*^{N+3}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} \, h \left(\left|\theta - \theta'\right|^{2\gamma-3} + \left|\theta - \theta'\right|^{3\gamma-3}\right)
\end{aligned}$$

Let us turn to A_{k4}. By (2.39) we have the bound:

$$\begin{aligned}
|\mathcal{T}_h \partial_{\theta'} g_k| &\leq C \frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^N} \prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\widetilde{\alpha}_i+\beta_i} |\theta + h - \theta'|^{\gamma-1} \\
&\leq C \frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^N} \prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\widetilde{\alpha}_i+\beta_i} |\theta - \theta'|^{\gamma-1}
\end{aligned}$$

where we used (2.31) in the last inequality. Thus,

$$\begin{aligned}
|A_{k4}| &\leq |\triangle_h (\partial_\theta \phi_k)| \, |\mathcal{T}_h \partial_{\theta'} g_k| \\
&\leq C \frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^3}{|\boldsymbol{Z}_0|_*^{N+3}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} \left(h^\gamma |\theta - \theta'|^{\gamma-2} + h \left|\theta - \theta'\right|^{2\gamma-3}\right).
\end{aligned}$$

We may now combine our estimates on $A_{k1}, \ldots, A_{k4}$ to obtain the bound on $\triangle_h \partial'_\theta A_k$ noting that the bound is dominated by $h^\gamma |\theta - \theta'|^{-2}$ and $h|\theta - \theta'|^{\gamma-3}$. The bounds on $\triangle_h \partial'_\theta A_0$ as well as $\triangle_h \partial'_\theta B_k, k = 0, \ldots, n$ can be obtained in the same fashion. This concludes the proof. $\square$

The class $\mathcal{S}^H_{n,k,\gamma}$ enjoys some mapping properties when acted on by different derivative operators. Indeed,

Lemma 2.2.5. *If $g \in \mathcal{S}^H_{k,n,\gamma}$, then $\partial_t g$ is a finite sum of functions $f_i \in \mathcal{S}^H_{k-1,n,\gamma}$.*

Proof. It is enough to show that a single function of the form

$$g(t,\theta,\theta') = \prod_{i=0}^{n} \left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_i} \left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_i}$$

for functions $\{\boldsymbol{Z}_i\}_{i=0}^n$ with $\boldsymbol{Z}_i = (Z_i(t,\theta), W_i(t,\theta)) \in C^k(I; C^{n,\gamma})$, $|\boldsymbol{Z}_0|_* > 0$ has the desired mapping property. Further, since

$$\partial_t g(t,\theta,\theta') = \sum_{j=0}^{n} \partial_t \left(\left(\frac{\Delta Z_j}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_j} \left(\frac{\Delta W_j}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_j}\right) \prod_{i=0, i\neq j}^{n} \left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_i} \left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_i},$$

it suffices to show that any function of the form

$$\phi(t,\theta,\theta') = \frac{\Delta V}{|\Delta \boldsymbol{Z}_0|}$$

has $\partial_t \phi \in \mathcal{S}^H_{k-1,n,\gamma}$ for any function $V \in C^k(I; C^{n,\gamma})$. But,

$$\partial_t \phi(t,\theta,\theta') = \frac{\Delta \partial_t V}{|\Delta \boldsymbol{Z}_0|} - \frac{\Delta V \Delta \boldsymbol{Z}_0 \cdot \Delta \partial_t \boldsymbol{Z}_0}{|\Delta \boldsymbol{Z}_0|^3}.$$

Because $V, \boldsymbol{Z}_0 \in C^k(I; C^{n,\gamma})$, we have $\partial_t V, \partial_t \boldsymbol{Z}_0 \in C^{k-1}(I; C^{n,\gamma})$, so $\partial_t \phi$ is a sum of functions in $\mathcal{S}^H_{k-1,n,\gamma}$ as desired. □

In what follows, we will need to make sense of Gâteaux derivatives of this class of functions. To that end, we prove the following lemma.

Lemma 2.2.6. *The Gâteaux derivative of any $g \in \mathcal{S}^H_{k,n,\gamma}$ about one of the component functions $\boldsymbol{Z}_i$ in direction $\boldsymbol{V} \in C^k(I; C^{n,\gamma})$ is a finite sum of functions f_k such that each $f_k \in \mathcal{S}^H_{k,n,\gamma}$. Let n be the number of component functions $\boldsymbol{Z}_i$ g has and let the exponents be α_i, β_i, with $N_0 = \sum_{i=0}^{n} \alpha_i + \beta_i$ and $N = \sum_{i=1}^{n} \alpha_i + \beta_i$ as defined in lemma 2.2.3. Define $\boldsymbol{Z}_{n+1} = \boldsymbol{V}$ and*

$\tilde{N} = \sum_{i=1}^{n+1} \tilde{\alpha}_i + \tilde{\beta}_i$ *and* $\tilde{N}_0 = \sum_{i=1}^{n+1}$. *Using these, each term* f_k *in* g*'s Gâteaux derivative has the following:*

(i) If $i \neq 0$*, the exponents* α_j*,* β_j *are unchanged unless* $j = i$ *and in that case the new exponents* $\tilde{\alpha}_i = \alpha_i - 1$ *and* $\tilde{\beta}_i = \beta_i - 1$*. Also,* $\tilde{N} = N$ *and* $\tilde{N}_0 = N_0$*.*

(ii) If $i = 0$*, then the exponents are as follows: if* $j \neq i$ *then* α_j*,* β_j *are unchanged and one of the following is true*

- $\tilde{\alpha}_0 = \alpha_0 - 1$ *and* $\tilde{\beta}_0 = \beta_0$
- $\tilde{\alpha}_0 = \alpha_0$ *and* $\tilde{\beta}_0 = \beta_0 - 1$
- $\tilde{\alpha}_0 = \alpha_0 + 1$ *and* $\tilde{\beta}_0 = \beta_0$
- $\tilde{\alpha}_0 = \alpha_0$ *and* $\tilde{\beta}_0 = \beta_0 + 1$*.*

In general, $\tilde{N} = N + 1$ *and* $\tilde{N}_0 \leq N_0 + 2$

Proof. Let g be of the form

$$g(t, \theta, \theta') = \prod_{i=0}^{n} \left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|} \right)^{\alpha_i} \left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|} \right)^{\beta_i}.$$

Without loss of generality, we will linearize about $\boldsymbol{Z}_0 = (Z_0, W_0)$. Consider $\boldsymbol{V} = (U, V) \in C^k(I; C^{n,\gamma})$ such that $|\boldsymbol{Z}_0 + \epsilon_0 \boldsymbol{V}|_* > 0$ for some $\epsilon_0 > 0$. Then, for all $\epsilon < \epsilon_0$,

$$\begin{aligned} \mathcal{L}g = {} & \frac{d}{d\epsilon} \left(\left(\frac{\Delta(Z_0 + \epsilon U)}{|\Delta(\boldsymbol{Z}_0 + \epsilon \boldsymbol{V})|} \right)^{\alpha_0} \left(\frac{\Delta(W_0 + \epsilon V)}{|\Delta(\boldsymbol{Z}_0 + \epsilon \boldsymbol{V})|} \right)^{\beta_0} \right) \Bigg|_{\epsilon=0} \prod_{i=1}^{n} \left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|} \right)^{\alpha_i} \left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|} \right)^{\beta_i} \\ & + \left(\frac{\Delta Z_0}{|\Delta \boldsymbol{Z}_0|} \right)^{\alpha_0} \left(\frac{\Delta W_0}{|\Delta \boldsymbol{Z}_0|} \right)^{\beta_0} \frac{d}{d\epsilon} \left(\prod_{i=1}^{n} \left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|} \right)^{\alpha_i} \left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|} \right)^{\beta_i} \right) \Bigg|_{\epsilon=0} =: A + B. \end{aligned}$$

We compute,

$$\begin{aligned} A = {} & \alpha_0 \left(\frac{\Delta Z_0}{|\Delta \boldsymbol{Z}_0|} \right)^{\alpha_0 - 1} \left(\frac{\Delta U}{|\Delta \boldsymbol{Z}_0|} - \frac{\Delta Z_0 \Delta \boldsymbol{Z}_0 \cdot \Delta \boldsymbol{V}}{|\Delta \boldsymbol{Z}_0|^3} \right) \left(\frac{\Delta W_0}{|\Delta \boldsymbol{Z}_0|} \right)^{\beta_0} m(\theta, \theta') \\ & + \beta_0 \left(\frac{\Delta W_0}{|\Delta \boldsymbol{Z}_0|} \right)^{\beta_0 - 1} \left(\frac{\Delta V}{|\Delta \boldsymbol{Z}_0|} - \frac{\Delta W_0 \Delta \boldsymbol{Z}_0 \cdot \Delta \boldsymbol{V}}{|\Delta \boldsymbol{Z}_0|^3} \right) \left(\frac{\Delta Z_0}{|\Delta \boldsymbol{Z}_0|} \right)^{\alpha_0} m(\theta, \theta'), \end{aligned}$$

where

$$m(\theta,\theta',1,n) := \prod_{i=1}^{n}\left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_i}\left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_i}.$$

Clearly A is in $\mathcal{S}^H_{k,n,\gamma}$. Similarly,

$$\begin{aligned}B = &\sum_{i=1}^{n}\alpha_i\left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_i-1}\left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_i}\left(\frac{\Delta Z_i \Delta \boldsymbol{Z}_0\cdot\Delta\boldsymbol{V}}{|\Delta \boldsymbol{Z}_0|^3}\right) l(\theta,\theta';i)\\ &+\sum_{i=1}^{n}\beta_i\left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_i}\left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_i-1}\left(\frac{\Delta W_i \Delta \boldsymbol{Z}_0\cdot\Delta\boldsymbol{V}}{|\Delta \boldsymbol{Z}_0|^3}\right) l(\theta,\theta';i),\end{aligned}$$

with

$$l(\theta,\theta';i) := \prod_{j=0,j\neq i}^{n}\left(\frac{\Delta Z_j}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_j}\left(\frac{\Delta W_j}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_j}.$$

B is also clearly of class $\mathcal{S}^H_{k,n,\gamma}$ and so $\mathcal{L}g \in \mathcal{S}^H_{k,n,\gamma}$. If we set $\boldsymbol{V} = \boldsymbol{Z}_{n+1}$ then we can rewrite term A as

$$\begin{aligned}\frac{1}{\alpha_0}A = &\left(\frac{\Delta Z_0}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_0-1}\left(\frac{\Delta W_0}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_0}\left(\frac{\Delta Z_{n+1}}{|\Delta \boldsymbol{Z}_0|}\right)\prod_{i=1}^{n}\left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_i}\left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_i}\\ &-\left(\frac{\Delta Z_0}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_0+1}\left(\frac{\Delta W_0}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_0}\left(\frac{\Delta Z_{n+1}}{|\Delta \boldsymbol{Z}_0|}\right)\prod_{i=1}^{n}\left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_i}\left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_i}\\ &-\left(\frac{\Delta Z_0}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_0}\left(\frac{\Delta W_0}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_0+1}\left(\frac{\Delta W_{n+1}}{|\Delta \boldsymbol{Z}_0|}\right)\prod_{i=1}^{n}\left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_i}\left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_i}\\ &+\left(\frac{\Delta Z_0}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_0}\left(\frac{\Delta W_0}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_0-1}\left(\frac{\Delta W_{n+1}}{|\Delta \boldsymbol{Z}_0|}\right)\prod_{i=1}^{n}\left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_i}\left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_i}\\ &-\left(\frac{\Delta Z_0}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_0+1}\left(\frac{\Delta W_0}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_0}\left(\frac{\Delta Z_{n+1}}{|\Delta \boldsymbol{Z}_0|}\right)\prod_{i=1}^{n}\left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_i}\left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_i}\\ &-\left(\frac{\Delta Z_0}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_0}\left(\frac{\Delta W_0}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_0+1}\left(\frac{\Delta W_{n+1}}{|\Delta \boldsymbol{Z}_0|}\right)\prod_{i=1}^{n}\left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_i}\left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_i}.\end{aligned}$$

We can do the same for term B and find

$$
\begin{aligned}
B = &\sum_{i=1}^{n} \alpha_i \left(\frac{\Delta Z_0}{|\Delta \boldsymbol{Z}_0|}\right)\left(\frac{\Delta Z_{n+1}}{|\Delta \boldsymbol{Z}_0|}\right)\prod_{j=0}^{n}\left(\frac{\Delta Z_j}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_j}\left(\frac{\Delta W_j}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_j} \\
&+ \sum_{i=1}^{n} \alpha_i \left(\frac{\Delta W_0}{|\Delta \boldsymbol{Z}_0|}\right)\left(\frac{\Delta W_{n+1}}{|\Delta \boldsymbol{Z}_0|}\right)\prod_{j=0}^{n}\left(\frac{\Delta Z_j}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_j}\left(\frac{\Delta W_j}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_j} \\
&+ \sum_{i=1}^{n} \beta_i \left(\frac{\Delta Z_0}{|\Delta \boldsymbol{Z}_0|}\right)\left(\frac{\Delta Z_{n+1}}{|\Delta \boldsymbol{Z}_0|}\right)\prod_{j=0}^{n}\left(\frac{\Delta Z_j}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_j}\left(\frac{\Delta W_j}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_j} \\
&+ \sum_{i=1}^{n} \beta_i \left(\frac{\Delta W_0}{|\Delta \boldsymbol{Z}_0|}\right)\left(\frac{\Delta W_{n+1}}{|\Delta \boldsymbol{Z}_0|}\right)\prod_{j=0}^{n}\left(\frac{\Delta Z_j}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_j}\left(\frac{\Delta W_j}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_j}.
\end{aligned}
$$

We can inspect the terms above and note the changes in each of the exponents α_j, β_j is as stated.

□

Next, we show that we may exchange derivatives in θ on $g \in \mathcal{S}^H_{k,n,\gamma}$ for those in θ' by adding in a correcting function from a less smooth class $\mathcal{S}^H_{k,n-1,\gamma}$, or vice versa.

Lemma 2.2.7. *If $g \in \mathcal{S}^H_{k,n,\gamma}$ for $n \geq 1$ then $\partial_\theta g = -\partial_{\theta'} g + f$ for some $f \in \mathcal{S}^H_{k,n-1,\gamma}$.*

Proof. By linearity and the power and product rules, it suffices to show that this is true for a function

$$
\phi = \frac{\Delta V}{|\Delta \boldsymbol{Z}_0|}
$$

for $V, \boldsymbol{Z}_0 \in C^k(I; C^{n,\gamma})$ with $|\boldsymbol{Z}_0|_* > 0$. We take a derivative in θ. Then,

$$
\begin{aligned}
\partial_\theta \phi &= \frac{\partial_\theta V}{|\Delta \boldsymbol{Z}_0|} - \frac{\Delta V \Delta \boldsymbol{Z}_0 \cdot \partial_\theta \boldsymbol{Z}_0}{|\Delta \boldsymbol{Z}_0|^3} \\
&= \frac{\partial_{\theta'} V'}{|\Delta \boldsymbol{Z}_0|} - \frac{\Delta V \Delta \boldsymbol{Z}_0 \cdot \partial_{\theta'} \boldsymbol{Z}_0'}{|\Delta \boldsymbol{Z}_0|^3} + \frac{\partial_\theta V - \partial_{\theta'} V'}{|\boldsymbol{Z}_0|} - \frac{\Delta V \Delta \boldsymbol{Z}_0 \cdot (\partial_\theta \boldsymbol{Z}_0 - \partial_{\theta'} \boldsymbol{Z}_0')}{|\Delta \boldsymbol{Z}_0|^3} \\
&= -\partial_{\theta'} \phi + \frac{\partial_\theta V - \partial_{\theta'} V'}{|\boldsymbol{Z}_0|} - \frac{\Delta V \Delta \boldsymbol{Z}_0 \cdot (\partial_\theta \boldsymbol{Z}_0 - \partial_{\theta'} \boldsymbol{Z}_0')}{|\Delta \boldsymbol{Z}_0|^3}.
\end{aligned}
$$

Since $V, \boldsymbol{Z}_0 \in C^k(I; C^{n,\gamma})$ and $\partial_\theta V, \partial_\theta \boldsymbol{Z}_0 \in C^k(I; , C^{n-1,\gamma})$, we have

$$f := \frac{\partial_\theta V - \partial_{\theta'} V'}{|\boldsymbol{Z}_0|} - \frac{\Delta V \Delta \boldsymbol{Z}_0 \cdot (\partial_\theta \boldsymbol{Z}_0 - \partial_{\theta'} \boldsymbol{Z}_0')}{|\Delta \boldsymbol{Z}_0|^3} \in \mathcal{S}^H_{k,n-1,\gamma}$$

□

The class $\mathcal{S}^H_{k,n,\gamma}$ clearly addresses the tensor kernel in (2.2), leaving us to make sense of the logrithmic kernel (2.1). Consider a function f which is a linear combination of functions of the form

$$f(t,\theta,\theta') = \log|\Delta \boldsymbol{Z}|$$

for some $\boldsymbol{Z} \in C^k(I; C^{1,\gamma})$ with $|\boldsymbol{Z}|_* > 0$. Our actual evolution is dictated by a integral equation which features a kernel which is the derivative of the above function. In the nonlinear setting, we will need to linearize our equations of motion. As we linearize the main evolution equation (1.20) about the initial data, the Frechét derivative will either hit the kernel or it will hit the elastic forcing law. The linearization of the kernel results in a new kernel which is the derivative of a function in class $\mathcal{S}^H$. When the kernel is left alone, we can show that it is approximately the Hilbert Transform.

Lemma 2.2.8. *The Gâteaux derivative of any function f of the form*

$$f(t,\theta,\theta') = \log|\Delta \boldsymbol{Z}|$$

for any $\boldsymbol{Z} \in C^k(I; C^{n,\gamma})$ with $|\boldsymbol{Z}|_* > 0$ in direction $\boldsymbol{V} \in C^k(I; C^{n,\gamma})$ is of class $\mathcal{S}^H_{k,n,\gamma}$.

Proof. Let $\boldsymbol{Z}, \boldsymbol{V} \in C^k(I; C^{n,\gamma})$ with $|\boldsymbol{Z}|_* > 0$. Let $\epsilon_0 > 0$ such that $|\boldsymbol{Z} + \epsilon_0 \boldsymbol{V}|_* > 0$. For $\epsilon < \epsilon_0$, we compute,

$$\begin{aligned} \frac{d}{d\epsilon} \log|\Delta(\boldsymbol{Z} + \epsilon \boldsymbol{V})|\bigg|_{\epsilon=0} &= \frac{(\Delta \boldsymbol{Z} + \epsilon \Delta \boldsymbol{V}) \cdot \Delta \boldsymbol{V}}{|\Delta \boldsymbol{Z} + \epsilon \Delta \boldsymbol{V}|^2}\bigg|_{\epsilon=0} \\ &= \frac{\Delta \boldsymbol{Z} \cdot \Delta \boldsymbol{V}}{|\Delta \boldsymbol{Z}|^2} \end{aligned}$$

which is clearly in $\mathcal{S}^H_{k,n,\gamma}$. □

Before we prove the second result, we will need a few more estimates on functions of this type.

Lemma 2.2.9. *Suppose functions $\boldsymbol{Z}(\theta) = (Z(\theta), W(\theta))$ and $V(\theta)$ belong to $C^{1,\gamma}(\mathbb{S}^1)$. Assume also that $|\boldsymbol{Z}|_* > 0$. Let*

$$q(\theta, \theta') = \frac{\partial_{\theta'} V' - (\Delta V/(\theta - \theta'))}{|\Delta \boldsymbol{Z}|}.$$

(i) We have the following estimates:

$$\left|q(\theta, \theta')\right| \leq \frac{\|V\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*} \left|\theta - \theta'\right|^{\gamma - 1}. \tag{2.50}$$

$$\left|\partial_\theta q(\theta, \theta')\right| \leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*^2} \left|\theta - \theta'\right|^{\gamma - 2}. \tag{2.51}$$

(ii) Suppose $0 < h < |\theta - \theta' + h/2|$ and $0 < \theta + h < 2\pi$. Then, the following estimate holds.

$$\left|\triangle_h \partial_\theta q(\theta, \theta')\right| \leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^3} \left(h^\gamma \left|\theta - \theta'\right|^{-2} + h \left|\theta - \theta'\right|^{\gamma - 3}\right) \tag{2.52}$$

where the constant C depends only on γ.

Proof. We first prove (2.50). By (2.21) and (2.22) (applied to $\partial_{\theta'} V'$) we see that

$$|q| \leq \frac{|(\Delta V/(\theta - \theta')) - \partial_{\theta'} V'|}{|\Delta \boldsymbol{Z}/(\theta - \theta')|} \left|\theta - \theta'\right|^{-1} \leq \frac{\|V\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*} \left|\theta - \theta'\right|^{\gamma - 1}.$$

We next turn to (2.51).

$$\partial_\theta q = \left(\frac{\Delta V}{\theta - \theta'} - \partial_\theta V\right) \frac{1}{(\theta - \theta') \left|\Delta \boldsymbol{Z}\right|} - \left(\frac{\Delta V}{\theta - \theta'} - \partial_{\theta'} V'\right) \frac{\Delta \boldsymbol{Z} \cdot \partial_\theta \boldsymbol{Z}}{\left|\Delta \boldsymbol{Z}\right|^3} \tag{2.53}$$

Using (2.21) and (2.22), we have:

$$|\partial_\theta q| \leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*^2} \left|\theta - \theta'\right|^{\gamma - 2}.$$

To obtain estimates in $\triangle_h \partial_\theta q$, take the difference of the two terms in (2.53). This can be obtained in a similar fashion to the calculation in Lemma 2.2.2 to obtain (2.16). We omit the details. $\square$

Lemma 2.2.10. *Suppose we have a set of functions* $\{\boldsymbol{Z}_i\}_{i=0}^m$ *such that* $\boldsymbol{Z}_i \in C^{n,\gamma}$ *with* $|\boldsymbol{Z}_0|_* > 0$ *and some function* $f(\theta, \theta')$ *of the form*

$$f(\theta, \theta') = \frac{(\partial_{\theta'} V' - \Delta V/(\theta - \theta'))}{|\Delta \boldsymbol{Z}_0|} \prod_{i=0}^{m} \left(\frac{\Delta Z_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\alpha_i} \left(\frac{\Delta W_i}{|\Delta \boldsymbol{Z}_0|}\right)^{\beta_i}, \tag{2.54}$$

where $V = Z_k$ *or* $V = W_k$ *for some* $k = 0, \dots m$*. Let*

$$N = \sum_{i=1}^{m} (\alpha_i + \beta_i),\ N_0 = \sum_{i=0}^{n} (\alpha_i + \beta_i) = (\alpha_0 + \beta_0) + N.$$

Then, we have:

$$|f(\theta, \theta')| \le \frac{\|\boldsymbol{Z}_k\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+1}} \prod_{i=1}^{m} \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} |\theta - \theta'|^{\gamma-1}, \tag{2.55}$$

$$\left|\partial_\theta f(\theta, \theta')\right| \le C \frac{\|\boldsymbol{Z}_k\|_{C^{1,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^{m} \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} |\theta - \theta'|^{\gamma-2}, \tag{2.56}$$

where C *depends only on* γ*.*

Proof. We can write g as:

$$f = qg,\ q = \frac{(\partial_{\theta'} V' - \Delta V/(\theta - \theta'))}{|\Delta \boldsymbol{Z}_0|}. \tag{2.57}$$

Note that g is exactly of the same form as the function estimated in Lemma (2.2.3). Inequality (2.55) is a direct consequence of the (2.50) of the previous lemma and of (2.38). Inequality (2.55) is now immediate. We turn to (2.56). Note that:

$$|\partial_\theta g| \le |\partial_\theta q| \, |g| + |q| \, |\partial_\theta g| \, .$$

Each of the factors may be estimated using (2.51), (2.38), (2.50) and (2.47), from which we obtain the desired inequality. □

Lemma 2.2.11. *Let* $f(\theta, \theta')$ *be as in Lemma 2.2.10 and suppose* $0 < h < |\theta - \theta' + h/2|$ *and* $0 < \theta + h < 2\pi$*.*

$$\left|\triangle_h f(\theta, \theta')\right| \le C \frac{\|\boldsymbol{Z}_k\|_{C^{1,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^{n} \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} h|\theta - \theta'|^{\gamma-2} \tag{2.58}$$

$$\left|\triangle_h \partial_\theta f(\theta, \theta')\right| \le C \frac{\|\boldsymbol{Z}_k\|_{C^{1,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}_0|_*^{N+3}} \prod_{i=1}^{n} \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} \left(h^\gamma \left|\theta - \theta'\right|^{-2} + h \left|\theta - \theta'\right|^{\gamma-3}\right), \tag{2.59}$$

where the constant C *does not depends only on* γ*.*

Proof. Express f as in (2.57). The bound (2.58) is a simple consequence of (2.56), and can be obtained in the same way as we obtained (2.47) from (2.39) in the proof of Lemma 2.2.4.

To obtain (2.59), we write $\triangle_h \partial_\theta f(\theta, \theta')$ as:

$$\begin{aligned}\triangle_h \partial_\theta f(\theta, \theta') &= B_1 + B_2 + B_3 + B_4,\\ B_1 &= (\triangle_h q)(\mathcal{T}_h \partial_\theta g),\ B_2 = q(\triangle_h \partial_\theta g),\\ B_3 &= (\triangle_h \partial_\theta q)(\mathcal{T}_h g),\ B_4 = (\partial_\theta q)(\triangle_h g).\end{aligned}$$

We estimate each term. First note that

$$\begin{aligned}|\triangle_h q| \leq h \int_0^1 \left|\partial_\theta q(\theta + sh, \theta')\right| ds &\leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^2} \int_0^1 \left|\theta + sh - \theta'\right|^{\gamma-2} ds\\ &\leq C \frac{\|V\|_{C^{1,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^2} h \left|\theta - \theta'\right|^{\gamma-2}.\end{aligned}$$

Using (2.39) and (2.31), we have:

$$|\mathcal{T}_h \partial_\theta g| \leq C \frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+1}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} |\theta - \theta'|^{\gamma-1}$$

Thus,

$$|B_1| \leq C \frac{\|\boldsymbol{Z}_k\|_{C^{1,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}_0|_*^{N+3}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} h|\theta - \theta'|^{2\gamma-3}$$

To obtain B_2, we must estimate $\triangle_h \partial_\theta g$. Let us use the notation in (2.44). Consider A_l and write this as $A_l = (\partial_\theta \phi_l) g_l$ (as in (2.46)). We have:

$$\triangle_h A_l = (\triangle_h \partial_\theta \phi_l)(\mathcal{T}_h g_l) + \partial_\theta \phi_l (\triangle_h g_l).$$

Using (2.16) and (2.38), we have:

$$|\triangle_h \partial_\theta \phi_l| \, |\mathcal{T}_h g_l| \leq C \frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} \left(h^\gamma \left|\theta - \theta'\right|^{-1} + h \left|\theta - \theta'\right|^{\gamma-2}\right)$$

Using (2.49) and (2.11), we have:

$$|\partial_\theta \phi_l| \, |\triangle_h g_l| \leq C \frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} h \left|\theta - \theta'\right|^{2\gamma-2}$$

Combining the above with (2.50), we have:

$$\begin{aligned}&|B_2| \leq\\ &\quad C \frac{\|\boldsymbol{Z}_k\|_{C^{1,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}_0|_*^{N+3}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} \left(h \left|\theta - \theta'\right|^{3\gamma-3} + h^\gamma \left|\theta - \theta'\right|^{\gamma-2} + h \left|\theta - \theta'\right|^{2\gamma-3}\right)\end{aligned}$$

Let us consider B_3. Using (2.52) and (2.38), we have:

$$\begin{aligned}|B_3| &= |\triangle_h \partial_\theta q|\,|\mathcal{T}_h g| \\ &\leq C\frac{\|\boldsymbol{Z}_k\|_{C^{1,\gamma}}\,\|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|^3_*}\prod_{i=1}^{n}\|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}\left(h^\gamma\left|\theta-\theta'\right|^{-2}+h\left|\theta-\theta'\right|^{\gamma-3}\right).\end{aligned}$$

Finally, using (2.51) and (2.49) (as applied to g), we have:

$$|B_4| \leq C\frac{\|\boldsymbol{Z}_k\|_{C^{1,\gamma}}\,\|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|^{N+3}_*}\prod_{i=1}^{n}\|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}\,h\left|\theta-\theta'\right|^{2\gamma-3}.$$

Combining the estimates for $B_k, k = 1, \ldots, 4$ and taking only the leading order terms, we obtain the desired estimate. □

2.3 Estimates on Integral Operators

Now that we have built up a collection of estimates on the kernels of interest, we can now prove how operators featuring them act,

Proposition 2.3.1. *Any integral operator of the form*

$$Tu := \int_{\mathbb{S}^1}\left(\partial_{\theta'}g(\theta,\theta')\right)u'd\theta'$$

for some $g \in \mathcal{S}^H_{0,1,\gamma}$ with functions $\{\boldsymbol{Z}_i\}_{i=0}^n$ for any $\gamma \in (0,1)$ enjoys the following mapping properties:

i.) if $\gamma \neq 1/2$ then

$$\|Tu\|_{C^{\lfloor 2\gamma\rfloor,2\gamma-\lfloor 2\gamma\rfloor}} \leq C\frac{\|\boldsymbol{Z}_0\|^3_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|^{N+3}_*}\prod_{i=1}^{n}\|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}\|u\|_{C^{0,\gamma}}.$$

ii.) if $\gamma = 1/2$ then for any $\alpha \in (0,1)$,

$$\|Tu\|_{C^{0,\alpha}} \leq C\frac{\|\boldsymbol{Z}_0\|^3_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|^{N+3}_*}\prod_{i=1}^{n}\|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}\|u\|_{C^{0,\gamma}}.$$

The above constants C depend only on γ, the constants multiplying the terms in g and N_0 as defined in lemma 2.2.3.

Proof. Let us start with the case that $\gamma < 1/2$. Let $u \in C^\gamma$ and $g \in \mathcal{S}^H_{0,1,\gamma}$. Then, using lemma 2.2.3, we have

$$
\begin{aligned}
|Tu| &\leq \int_{\mathbb{S}^1} \left|\partial_{\theta'} g(\theta,\theta')\right| \left|u'\right| d\theta' \leq C \frac{\|u\|_{C^{0,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+1}} \prod_{i=1}^n \|Z_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} \int_{\mathbb{S}^1} \left|\theta-\theta'\right|^{\gamma-1} d\theta' \\
&\leq C \frac{\|u\|_{C^{0,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+1}} \prod_{i=1}^n \|Z_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i},
\end{aligned}
$$

where C is a function only of γ, N_0 and coefficients of the various terms of g as a linear combination of quotients of differences. Note that since the kernel of T is a perfect derivative, T maps any constant to zero. And thus, we may choose any constant value c and write

$$
Tu = \int_{\mathbb{S}^1} \partial_{\theta'} g(\theta,\theta') \left(u' - c\right) d\theta'.
$$

Now, without loss of generality, let $\eta > 0$. Then,

$$
\triangle_\eta Tu = \int_{\mathbb{S}^1} \left(\partial_{\theta'} g(\theta+\eta,\theta') - \partial_{\theta'} g(\theta,\theta')\right) \left(u(\theta') - u(\theta)\right) d\theta',
$$

where here we have chosen our constant $c = u(\theta)$. Now, we divide our domain of integration into two sub domains $\mathcal{I}_s$ and $\mathcal{I}_f$ as defined in (2.3). So,

$$
\begin{aligned}
\triangle_\eta Tu = &\int_{\mathcal{I}_s} \left(\partial_{\theta'} g(\theta+\eta,\theta') - \partial_{\theta'} g(\theta,\theta')\right) \left(u(\theta') - u(\theta)\right) d\theta' \\
&+ \int_{\mathcal{I}_f} \left(\partial_{\theta'} g(\theta+\eta,\theta') - \partial_{\theta'} g(\theta,\theta')\right) \left(u(\theta') - u(\theta)\right) d\theta' =: I_1 + I_2.
\end{aligned}
$$

On I_1, we have

$$\begin{aligned}
|I_1| &\le \int_{\mathcal{I}_s} \left|\partial_{\theta'} g(\theta+\eta,\theta') - \partial_{\theta'} g(\theta,\theta')\right| \left|u-u'\right| d\theta' \\
&\le \|u\|_{C^{0,\gamma}} \int_{\mathcal{I}_s} \left(\left|\partial_{\theta'} g(\theta+\eta,\theta')\right| + \left|\partial_{\theta'} g(\theta,\theta')\right|\right) \left|\theta-\theta'\right|^{\gamma} d\theta' \\
&\le C\eta^{\gamma} \frac{\|u\|_{C^{0,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+1}} \prod_{i=1}^{n} \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} \int_{\mathcal{I}_s} \left|\theta+\eta-\theta'\right|^{\gamma-1} + \left|\theta-\theta'\right|^{\gamma-1} d\theta' \\
&\le C\eta^{2\gamma} \frac{\|u\|_{C^{0,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+1}} \prod_{i=1}^{n} \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i},
\end{aligned}$$

where in the above we used estimate (2.39) from lemma 2.2.3 and C depends only on γ, N_0 and the constants multiplying the terms in g. As for term I_2, for any $\theta' \in \mathcal{I}_f$, estimate (2.39) implies

$$\left|\triangle_\eta \partial_{\theta'} g(\theta,\theta')\right| \le C \frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^{n} \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} \, \eta \left|\theta-\theta'\right|^{\gamma-2}$$

for constant C depending only on γ and the constants multiplying the terms of g. Using this,

$$\begin{aligned}
|I_2| &\le C\eta \frac{\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^{n} \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} \int_{\mathcal{I}_f} \left|\theta-\theta'\right|^{\gamma-2} \left|u-u'\right| d\theta' \\
&\le C\eta \frac{\|u\|_{C^{0,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^{n} \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} \int_{\mathcal{I}_f} \left|\theta-\theta'\right|^{2\gamma-2} d\theta' \\
&\le C\eta \frac{\|u\|_{C^{0,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^{n} \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i} \int_{\eta/2}^{2\pi} \left|\theta-\theta'\right|^{2\gamma-2} d\theta' \\
&\le C\eta^{2\gamma} \frac{\|u\|_{C^{0,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^{n} \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i}
\end{aligned}$$

for C depending only on γ and the constants multiplying the terms of g. Combining terms I_1 and I_2 yields

$$|\triangle_\eta T u| \le C\eta^{2\gamma} \frac{\|u\|_{C^{0,\gamma}} \|\boldsymbol{Z}_0\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^{n} \|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i}.$$

Thus,

$$\|Tu\|_{C^{0,2\gamma}} \leq C\frac{\|u\|_{C^{0,\gamma}}\,\|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}}\prod_{i=1}^n \|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}$$

since $\|\boldsymbol{Z}_0\|_{C^{1,\gamma}} \geq |\boldsymbol{Z}_0|_*$. Note here at if $\gamma = 1/2$, then $2\gamma = 1$. However, $\gamma = 1$ is not a Hölder space, but corresponds to Lipschitz continuous functions which are embedded in any Hölder continuous space with $0 < \alpha < 1$. It

For the case where $\gamma > 1/2$, as in the $\gamma < 1/2$ case, we find

$$|Tu| \leq C\frac{\|u\|_{C^{0,\gamma}}\,\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+1}}\prod_{i=1}^n \|Z_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}.$$

Since $2\gamma > 1$, we will show that $Tu \in C^{1,2\gamma-1}$. The derivative of the kernel of T behaves like $|\theta - \theta'|^{\gamma-2}$ as shown in (2.40) and so we cannot express the derivative of Tu as the integral operator with kernel $\partial_\theta\partial_{\theta'}g$ as it is too singular. However, we will show that $\partial_\theta Tu$ exists and is equal to

$$(\mathcal{A}u)(\theta) = \int_{\mathbb{S}^1} \partial_\theta\partial_{\theta'}g(\theta,\theta')\left(u(\theta') - u(\theta)\right)d\theta'.$$

First, note that this integral is well defined since (2.40) gives the estimate

$$\left|\partial_\theta\partial_{\theta'}g(\theta,\theta')\right| \leq C\frac{\|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}}\prod_{i=1}^n \|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}\left|\theta-\theta'\right|^{\gamma-2}$$

so that

$$\begin{aligned}
|\mathcal{A}u(\theta)| &\leq C\frac{\|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}}\prod_{i=1}^n \|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}\int_{\mathbb{S}^1}\left|\theta-\theta'\right|^{\gamma-2}\left|u(\theta)-u(\theta')\right|d\theta' \\
&\leq C\frac{\|u\|_{C^{0,\gamma}}\,\|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}}\prod_{i=1}^n \|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}\int_{\mathbb{S}^1}\left|\theta-\theta'\right|^{2\gamma-2}d\theta' \\
&\leq C\frac{\|u\|_{C^{0,\gamma}}\,\|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}}\prod_{i=1}^n \|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}
\end{aligned}$$

since $2\gamma - 1 > 0$.

Now, we show that the derivative of Tu is in fact $\mathcal{A}u$. Consider

$$\begin{aligned}\eta^{-1}\triangle_\eta(Tu)(\theta)-(\mathcal{A}u)(\theta)&=\int_{\mathbb{S}^1}\left(\eta^{-1}\triangle_\eta\partial_{\theta'}g(\theta,\theta')-\partial_\theta\partial_{\theta'}g(\theta,\theta')\right)\left(u(\theta')-u(\theta)\right)d\theta'\\&=\int_{\mathcal{I}_s}\left(\eta^{-1}\triangle_\eta\partial_{\theta'}g(\theta,\theta')-\partial_\theta\partial_{\theta'}g(\theta,\theta')\right)\left(u(\theta')-u(\theta)\right)d\theta'\\&+\int_{\mathcal{I}_f}\left(\eta^{-1}\triangle_\eta\partial_{\theta'}g(\theta,\theta')-\partial_\theta\partial_{\theta'}g(\theta,\theta')\right)\left(u(\theta')-u(\theta)\right)d\theta'\\&=:I_1+I_2,\end{aligned}$$

where we have made use of the fact that Tu annihilates constant $u(\theta)$. Now, consider I_1. Using lemma 2.2.3 we have

$$\begin{aligned}|I_1|&\le\int_{\mathcal{I}_s}\left(\eta^{-1}\left(\left|\partial_{\theta'}g(\theta,\theta')\right|+\left|\partial_{\theta'}g(\theta+\eta,\theta')\right|\right)+\left|\partial_\theta\partial_{\theta'}g(\theta,\theta')\right|\right)\|u\|_{C^{0,\gamma}}\left|\theta-\theta'\right|^\gamma d\theta'\\&\le C\frac{\|u\|_{C^{0,\gamma}}\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+1}}\prod_{i=1}^n\|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i}\int_{\mathcal{I}_s}\eta^{\gamma-1}\left(|\theta-\theta'|^{\gamma-1}+|\theta+\eta-\theta'|^{\gamma-1}\right)d\theta'\\&+C\frac{\|u\|_{C^{0,\gamma}}\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+1}}\prod_{i=1}^n\|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i}\int_{\mathcal{I}_s}|\theta-\theta'|^{2\gamma-2}d\theta'\\&\le C\frac{\|u\|_{C^{0,\gamma}}\|\boldsymbol{Z}_0\|_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+1}}\prod_{i=1}^n\|\boldsymbol{Z}_i\|_{C^{1,\gamma}}^{\alpha_i+\beta_i}\eta^{2\gamma-1},\end{aligned}$$

where we also used the fact that $2\gamma-1>0$. For term I_2 we estimate

$$\begin{aligned}\left|\eta^{-1}\triangle_\eta\partial_{\theta'}g(\theta,\theta')-\partial_\theta\partial_{\theta'}g(\theta,\theta')\right|&\le\int_0^1\left|\partial_\theta\partial_{\theta'}g(\theta+s\eta,\theta')-\partial_\theta\partial_{\theta'}g(\theta,\theta')\right|ds\\&=\int_0^1\left|\triangle_{s\eta}\partial_\theta\partial_{\theta'}g(\theta,\theta')\right|ds.\end{aligned}$$

Thus, with estimate (2.48) we find

$$\begin{aligned}
|I_2| &\leq \int_{\mathcal{I}_f} \left|\eta^{-1}\triangle_\eta \partial_{\theta'} g(\theta,\theta') - \partial_\theta\partial_{\theta'} g(\theta,\theta')\right| \|u\|_{C^{0,\gamma}} |\theta-\theta'|^\gamma d\theta' \\
&\leq C\frac{\|u\|_{C^{0,\gamma}} \|\boldsymbol{Z}_0\|^3_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+3}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}} \int_{\mathcal{I}_f} \eta^\gamma|\theta-\theta'|^{-2} + \eta|\theta-\theta'|^{\gamma-3}d\theta' \\
&\leq C\frac{\|u\|_{C^{0,\gamma}} \|\boldsymbol{Z}_0\|^3_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+3}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}} \eta^{2\gamma-1}.
\end{aligned}$$

Finally, since $2\gamma-1>0$ we conclude that

$$\partial_\theta(Tu)(\theta) = \lim_{\eta\to 0}\eta^{-1}\triangle_\eta Tu = \mathcal{A}u$$

as desired. Given the arguments above, we need only show that $[\partial_\theta Tu]_{C^{0,2\gamma-1}}$ is bounded. Let us assume that $2\pi>\eta>0$. Then,

$$\begin{aligned}
\triangle_\eta\partial_\theta Tu &= \int_{\mathbb{S}^1} \partial_\theta\partial_{\theta'} g(\theta+\eta,\theta')(u(\theta')-u(\theta+\eta))d\theta' \\
&\quad - \int_{\mathbb{S}^1} \partial_\theta\partial_{\theta'} g(\theta,\theta')(u(\theta')-u(\theta))d\theta' \\
&= \int_{\mathcal{I}_s} \partial_\theta\partial_{\theta'} g(\theta+\eta,\theta')(u(\theta')-u(\theta+\eta))d\theta' \\
&\quad + \int_{\mathcal{I}_s} \partial_\theta\partial_{\theta'} g(\theta,\theta')(u(\theta)-u(\theta'))d\theta' \\
&\quad + \int_{\mathcal{I}_f} \partial_\theta\partial_{\theta'} g(\theta,\theta')(u(\theta+h)-u(\theta))d\theta' \\
&\quad + \int_{\mathcal{I}_f} \triangle_\eta\partial_\theta\partial_{\theta'} g(\theta,\theta')(u(\theta+h)-u(\theta'))d\theta' \\
&=: I_1+I_2+I_3+I_4.
\end{aligned}$$

For I_1, using lemma 2.2.3 we have

$$\begin{aligned}
|I_1| &\leq C\frac{\|u\|_{C^{0,\gamma}} \|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}} \int_{\mathcal{I}_s} |\theta+\eta-\theta'|^{2\gamma-2}d\theta' \\
&\leq C\frac{\|u\|_{C^{0,\gamma}} \|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}} \prod_{i=1}^n \|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}} \eta^{2\gamma-1}.
\end{aligned}$$

The same argument holds for term I_2. For term I_3, we make use of lemma 2.2.3 again and

$$
\begin{aligned}
|I_3| &\leq C\eta^{\gamma}\frac{\|u\|_{C^{0,\gamma}}\|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}}\prod_{i=1}^{n}\|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}\int_{\mathcal{I}_f}|\theta-\theta'|^{\gamma-2}d\theta' \\
&\leq C\frac{\|u\|_{C^{0,\gamma}}\|\boldsymbol{Z}_0\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+2}}\prod_{i=1}^{n}\|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}\,\eta^{2\gamma-1}.
\end{aligned}
$$

Finally, for term I_4, we use estimate (2.48) from lemma 2.2.4 and find

$$
\begin{aligned}
|I_4| &\leq \int_{\mathcal{I}_f}\left|\triangle_\eta\partial_\theta\partial_{\theta'}g(\theta,\theta')\right|\left|u(\theta+\eta)-u(\theta')\right|d\theta' \\
&\leq C\frac{\|u\|_{C^{0,\gamma}}\|\boldsymbol{Z}_0\|^3_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+3}}\prod_{i=1}^{n}\|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}\int_{\mathcal{I}_f}\eta^{\gamma}|\theta-\theta'|^{\gamma-2}+\eta|\theta-\theta'|^{2\gamma-3}d\theta' \\
&\leq C\frac{\|u\|_{C^{0,\gamma}}\|\boldsymbol{Z}_0\|^3_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+3}}\prod_{i=1}^{n}\|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}\,\eta^{2\gamma-1}
\end{aligned}
$$

as desired. Thus, combining terms I_1 - I_4 we find

$$
[\partial_\theta(Tu)]_{C^{0,2\gamma-1}} \leq C\frac{\|u\|_{C^{0,\gamma}}\|\boldsymbol{Z}_0\|^3_{C^{1,\gamma}}}{|\boldsymbol{Z}_0|_*^{N+3}}\prod_{i=1}^{n}\|\boldsymbol{Z}_i\|^{\alpha_i+\beta_i}_{C^{1,\gamma}}\,.
$$

□

We also need to make sense of operators with the logrithmic kernel

$$
-\partial_{\theta'}\log|\Delta\boldsymbol{Z}|\,.
$$

For this purpose, we first show that the operator

$$
Lu = -\int_{\mathbb{S}^1}\partial_{\theta'}\left(\log|\Delta\boldsymbol{Z}|\right)u(\theta')d\theta'
$$

is approximately the Hilbert Transform. To do this, we will prove estimates on the operator

$$
L_C u = \int_{\mathbb{S}^1}\left(-\partial_{\theta'}\log|\Delta\boldsymbol{Z}| - \frac{1}{2}\cot\left(\frac{\theta-\theta'}{2}\right)\right)u(\theta')d\theta'.
$$

Proposition 2.3.2. *If $\boldsymbol{Z} \in C^{1,\gamma}(\mathbb{S}^1)$ with $\gamma \in (0,1)$ and $|\boldsymbol{Z}|_* > 0$, $u \in C^{0,\gamma}(\mathbb{S}^1)$ and*

(i) if $\gamma \neq 1/2$ then $L_C u \in C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}(\mathbb{S}^1)$ with

$$\|L_C u\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} \leq C \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^3}{|\boldsymbol{Z}|_*^3} \|u\|_{C^{0,\gamma}} .$$

(ii) if $\gamma = 1/2$ then $L_C u \in C^{0,\alpha}(\mathbb{S}^1)$ for any $\alpha \in (0,1)$ with

$$\|L_C u\|_{C^{0,\alpha}} \leq C \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^3}{|\boldsymbol{Z}|_*^3} \|u\|_{C^{0,\gamma}} .$$

In the above, the constant C does not depend on $\boldsymbol{Z}$ or u.

In fact, we have the following somewhat stronger bound:

$$\|L_C u\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} \leq C \frac{\|\partial_\theta \boldsymbol{Z}\|_{C^{0,\gamma}}^3}{|\boldsymbol{Z}|_*^3} \|u\|_{C^{0,\gamma}} \tag{2.60}$$

This is easily seen from the proof, but can also be seen from the fact that the kernel is invariant under translation. We also note that when $\gamma \leq 1/2$ we can reduce the exponents of $\|\boldsymbol{Z}\|_{C^{1,\gamma}}$ and $|\boldsymbol{Z}|_*$ by one power each.

Proof of Proposition 2.3.2. Let us first define

$$\begin{aligned} K_C(\theta, \theta') &:= -\partial_{\theta'} \log |\Delta \boldsymbol{Z}| - \frac{1}{2} \cot\left(\frac{\theta - \theta'}{2}\right) \\ &= \frac{\Delta \boldsymbol{Z} \cdot \partial_{\theta'} \boldsymbol{Z}'}{|\Delta \boldsymbol{Z}|^2} - \frac{1}{2} \cot\left(\frac{\theta - \theta'}{2}\right). \end{aligned}$$

We can rewrite K_C as

$$\begin{aligned} K_C(\theta, \theta') &= \left(\frac{\Delta \boldsymbol{Z} \cdot \partial_{\theta'} \boldsymbol{Z}'}{|\Delta \boldsymbol{Z}|^2} - \frac{1}{\theta - \theta'}\right) + \left(\frac{1}{\theta - \theta'} - \frac{1}{2} \cot\left(\frac{\theta - \theta'}{2}\right)\right) \\ &=: K_L(\theta, \theta') + R_C(\theta - \theta'). \end{aligned} \tag{2.61}$$

Let $\boldsymbol{Z}(\theta) = (Z(\theta), W(\theta))$. Note that K_L can be written as:

$$K_L = (K_L^X + K_L^Y),\ K_L^X(\theta, \theta') = \frac{(Z'_\theta - (\Delta Z/(\theta - \theta')))\Delta Z}{(\Delta Z)^2 + (\Delta W)^2}, \tag{2.62}$$

where $\Delta Z = Z(\theta) - Z(\theta')$ and likewise for ΔW. K_L^Y is simply the expression obtained by swapping Z and W in K_L^X. Notice that K_L^X is of the form (2.54) in Lemma 2.2.10 with $V = Z, \boldsymbol{Z}_0 = \boldsymbol{Z}$ and $n = 0, \alpha_0 = 1, \beta_0 = 0$. We may thus apply inequality (2.55) of Lemma 2.2.10 to obtain the estimate:

$$|K_L^X(\theta,\theta')| \leq \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*} \left|\theta - \theta'\right|^{\gamma-1}.$$

for some constant $C > 0$. The same estimate holds for K_L^Y; indeed, K_L^Y is of the form (2.54) with $V = W, \boldsymbol{Z}_0 = \boldsymbol{Z}$ with $n = 0, \alpha_0 = 1, \beta_0 = 0$. Thus, returning to (2.62), we obtain the estimate:

$$\left|K_L(\theta,\theta')\right| \leq \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*} \left|\theta - \theta'\right|^{\gamma-1}.$$

As for R_C of (2.61), simple calculus shows that

$$\left|R_C(\theta - \theta')\right| \leq C|\theta - \theta'|,$$

for some constant C. Plugging this back into (2.61) gives

$$|K_C(\theta,\theta')| \leq C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*}|\theta - \theta'|^{\gamma-1}. \tag{2.63}$$

An important property of the operator L_C is that it annihilates constants. Since $K_C(\theta,\theta')$ is a perfect derivative, for any constant C we have

$$(L_C C)(\theta) = \int_{\mathbb{S}^1} \partial_{\theta'} \log\left(\frac{|\Delta \boldsymbol{Z}|}{|\sin((\theta - \theta')/2)|}\right) C d\theta' = 0 \tag{2.64}$$

by the positivity, periodicity and continuity of the argument of the logarithm. Note that positivity is a result of the assumption $|\boldsymbol{Z}|_* > 0$. In particular, we have

$$(L_C u)(\theta) = \int_{\mathbb{S}^1} K_C(\theta,\theta')u(\theta')d\theta' = \int_{\mathbb{S}^1} K_C(\theta,\theta')(u(\theta') - C)d\theta', \tag{2.65}$$

for any constant C.

We turn to statement (i). We start with the case where $\gamma < 1/2$. Note that

$$|(L_C u)(\theta)| \leq \int_{\mathbb{S}^1} \left|K_C(\theta,\theta')u(\theta')\right| d\theta' \leq \|u\|_{C^0} \int_{\mathbb{S}^1} \left|K_C(\theta,\theta')\right| d\theta' \leq C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*} \|u\|_{C^0}, \tag{2.66}$$

where we used (2.63) and C does not depend on u or $\boldsymbol{Z}$. We thus see that $L_C u$ is bounded and thus well-defined. We must bound the seminorm $[L_C u]_{C^{0,2\gamma}}$. Consider the difference between two points, θ and $\theta+\eta$. Then,

$$\begin{aligned}\triangle_\eta(L_C u) &= \int_{\mathbb{S}^1} \left(K_C(\theta+\eta,\theta') - K_C(\theta,\theta')\right)(u(\theta') - u(\theta))d\theta' \\ &= \int_{\mathcal{I}_s} (\triangle_\eta K_C)(u(\theta')-u(\theta))d\theta' \\ &+ \int_{\mathcal{I}_f} (\triangle_\eta K_C)(u(\theta')-u(\theta))d\theta' =: A+B,\end{aligned}$$

where we used (2.65) with $C=u(\theta)$. We start by bounding term A. Using bound (2.63) on $\mathcal{I}_s$ we have,

$$\begin{aligned}|A| &\leq C\int_{\mathcal{I}_s} |\triangle_\eta K_C| \, \|u\|_{C^{0,\gamma}} |\theta-\theta'|^\gamma d\theta' \leq C\eta^\gamma \|u\|_{C^{0,\gamma}} \int_{\mathcal{I}_s} (|\mathcal{T}_\eta K_C| + |K_C|)d\theta' \\ &\leq C\eta^\gamma \|u\|_{C^{0,\gamma}} \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*} \int_{\mathcal{I}_s} \left(\frac{1}{|\theta+\eta-\theta'|^{1-\gamma}} + \frac{1}{|\theta-\theta'|^{1-\gamma}}\right)d\theta' \\ &\leq C\eta^{2\gamma} \|u\|_{C^{0,\gamma}} \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*}.\end{aligned} \tag{2.67}$$

In the above, the constants C may vary but do not depend on $\theta, \theta', u, \boldsymbol{Z}$ or η. Let us estimate B. From (2.58) of Lemma (2.2.11), for $\theta' \in \mathcal{I}_f$, we have:

$$|\triangle_\eta K_L(\theta,\theta')| \leq C\left(\frac{\|\boldsymbol{Z}\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}|^2_*}\right)\eta|\theta-\theta'|^{\gamma-2}.$$

Furthermore, it is easily seen that

$$\left|\triangle_\eta R_C(\theta-\theta')\right| \leq C\eta.$$

We thus have,

$$|\triangle_\eta K_C(\theta,\theta')| \leq C\left(\frac{\|\boldsymbol{Z}\|^2_{C^{1,\gamma}}}{|\boldsymbol{Z}|^2_*}\right)\eta|\theta-\theta'|^{\gamma-2}. \tag{2.68}$$

Hence,

$$
\begin{aligned}
|B| &\le C\int_{\mathcal{I}_f} |\triangle_\eta K_C| \left|u(\theta)-u(\theta')\right|) d\theta' \\
&\le \int_{\mathcal{I}_f} C\left(\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2}\eta|\theta-\theta'|^{\gamma-2}\right)\|u\|_{C^{0,\gamma}}|\theta-\theta'|^\gamma d\theta' \\
&\le C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2}\|u\|_{C^{0,\gamma}}\,\eta\int_{\mathcal{I}_f}|\theta-\theta'|^{2\gamma-2}d\theta' \\
&\le C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2}\|u\|_{C^{0,\gamma}}\,\eta\int_{\eta/2}^{2\pi}|s|^{2\gamma-2}ds \le C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2}\|u\|_{C^{0,\gamma}}\,\eta^{2\gamma}.
\end{aligned}
$$

Combining the bounds for A and B, we find that:

$$
|\triangle_\eta(L_c u)| \le C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2}\|u\|_{C^{0,\gamma}}\,\eta^{2\gamma}.
$$

This, together with (2.66) shows that $L_C u \in C^{0,2\gamma}(\mathbb{S}^1)$. Noting here that

$$
\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}}{|\boldsymbol{Z}|_*} \ge 1
$$

gives the desired bound.

We now consider the case where $\gamma > 1/2$. As in proposition 2.3.1, we must consider the derivative of $L_C u$. The derivative of kernel $\partial_\theta K_C$ behaves like $|\theta-\theta'|^{\gamma-2}$ per (2.70). So, like proposition 2.3.1m $\partial_\theta L_C u$ will be shown to equal

$$
(\mathcal{A}u)(\theta) = \int_{\mathbb{S}^1}\partial_\theta K_C(\theta,\theta')(u(\theta')-u(\theta))d\theta'. \tag{2.69}
$$

This integral is well-defined. The kernel $\partial_\theta K_C$ can be estimated by considering $\partial_\theta K_L$ and $\partial_\theta R_C$ separately. From (2.56), we have the bound

$$
\left|\partial_\theta K_L(\theta,\theta')\right| \le \frac{2\,\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2}\left|\theta-\theta'\right|^{\gamma-2}.
$$

Making use of the Taylor expansion of $\cot x$, it is easy to see that

$$
\left|\partial_\theta R_C(\theta-\theta')\right| \le C.
$$

Combining these gives

$$
\left|\partial_\theta K_C(\theta,\theta')\right| \le C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2}\left|\theta-\theta'\right|^{\gamma-2}. \tag{2.70}
$$

Thus,

$$
\begin{aligned}
|(\mathcal{A}u)(\theta)| &= \int_{\mathbb{S}^1} \left|\partial_\theta K_C(\theta,\theta')(u(\theta')-u(\theta))\right| d\theta' \\
&\le C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2}\|u\|_{C^{0,\gamma}}\int_{\mathbb{S}^1}\left|\theta-\theta'\right|^{2\gamma-2}d\theta' \le C\frac{\|\boldsymbol{X}\|_{C^{1,\gamma}}^2}{|\boldsymbol{X}|_*^2}\|u\|_{C^{0,\gamma}}.
\end{aligned}
\tag{2.71}
$$

where we used $2\gamma-1>0$ in the last inequality.

We now show that the derivative of $L_C u$ is indeed $\mathcal{A}u$. Consider the expression

$$
\begin{aligned}
\eta^{-1}\triangle_\eta(L_C u)(\theta)-(\mathcal{A}u)(\theta) &= \int_{\mathbb{S}^1}\left(\eta^{-1}\triangle_\eta K_C-\partial_\theta K_C\right)(\theta,\theta')(u(\theta')-u(\theta))d\theta' \\
&= I_1+I_2, \\
I_1 &= \int_{\mathcal{I}_s}\left(\eta^{-1}\triangle_\eta K_C-\partial_\theta K_C\right)(\theta,\theta')(u(\theta')-u(\theta))d\theta', \\
I_2 &= \int_{\mathcal{I}_f}\left(\eta^{-1}\triangle_\eta K_C-\partial_\theta K_C\right)(\theta,\theta')(u(\theta')-u(\theta))d\theta'.
\end{aligned}
$$

In the above, we used (2.64) or equivalently, (2.65). We estimate I_1 and I_2 separately. Assume $\eta>0$. For I_1, using (2.63) and (2.70) we have

$$
\begin{aligned}
|I_1| &\le C\int_{\mathcal{I}_s}(\eta^{-1}(|K_C|+|\mathcal{T}_\eta K_C|)+|\partial_\theta K_C|)\|u\|_{C^{0,\gamma}}\left|\theta-\theta'\right|^\gamma d\theta' \\
&\le C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2}\|u\|_{C^{0,\gamma}}\int_{\mathcal{I}_s}\left(\eta^{-1}\left(\left|\theta-\theta'\right|^{2\gamma-1}+\left|\theta+\eta-\theta'\right|^{2\gamma-1}\right)+\left|\theta-\theta'\right|^{2\gamma-2}\right)d\theta' \\
&\le C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2}\|u\|_{C^{0,\gamma}}\eta^{2\gamma-1}.
\end{aligned}
$$

To estimate I_2, note that, for $\theta'\in\mathcal{I}_f$:

$$
\begin{aligned}
\left|(\eta^{-1}\triangle_\eta K_C-\partial_\theta K_C)(\theta,\theta')\right| &= \int_0^1\left|\partial_\theta K_C(\theta+s\eta,\theta')-\partial_\theta K_C(\theta,\theta')\right|ds \\
&= \int_0^1|\triangle_{s\eta}\partial_\theta K_C|\,ds.
\end{aligned}
$$

Using (2.59) of Lemma 2.2.11, we have

$$
|\triangle_\eta\partial_\theta K_L| \le C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^3}{|\boldsymbol{Z}|_*^3}\left(\eta^\gamma\left|\theta-\theta'\right|^{-2}+\eta\left|\theta-\theta'\right|^{\gamma-3}\right).
$$

It is easily seen that this dominates $\triangle_\eta\partial_\theta R_C$ with a suitable constant. Therefore,

$$
|\triangle_\eta\partial_\theta K_C| \le C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^3}{|\boldsymbol{Z}|_*^3}\left(\eta^\gamma\left|\theta-\theta'\right|^{-2}+\eta\left|\theta-\theta'\right|^{\gamma-3}\right).
\tag{2.72}
$$

Thus,

$$
\begin{aligned}
|I_2| &\leq C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^3}{|\boldsymbol{Z}|_*^3}\|u\|_{C^{0,\gamma}}\int_{\mathcal{I}_f}\left(\eta^\gamma\left|\theta-\theta'\right|^{\gamma-2}+\eta\left|\theta-\theta'\right|^{2\gamma-3}\right)d\theta' \\
&\leq C\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^3}{|\boldsymbol{Z}|_*^3}\|u\|_{C^{0,\gamma}}\eta^{2\gamma-1}.
\end{aligned}
$$

Thus, since $2\gamma-1>0$,

$$
\partial_\theta L_C u=\lim_{\eta\to 0}\frac{1}{\eta}\triangle_\eta(L_C u)=\mathcal{A}u. \tag{2.73}
$$

We may now identify $\partial_\theta L_C u$ with the expression for $\mathcal{A}u$ in (2.69). We have only to bound the $C^{0,2\gamma-1}$ seminorm of $\partial_\theta L_C u$. Thus,

$$
\begin{aligned}
(\partial_\theta L_C u)(\theta+\eta)-(\partial_\theta L_C u)(\theta) &= \int_{\mathbb{S}^1}\partial_\theta K_C(\theta+\eta,\theta)(u(\theta')-u(\theta+\eta))d\theta' \\
&\quad -\int_{\mathbb{S}^1}\partial_\theta K_C(\theta,\theta')(u(\theta')-u(\theta))d\theta' \\
&= \int_{\mathcal{I}_s}\partial_\theta K_C(\theta,\theta')(u(\theta)-u(\theta'))d\theta' \\
&\quad +\int_{\mathcal{I}_s}\partial_\theta K_C(\theta+\eta,\theta')(u(\theta')-u(\theta+\eta))d\theta' \\
&\quad +\int_{\mathcal{I}_f}\partial_\theta K_C(\theta,\theta')(u(\theta)-u(\theta+\eta))d\theta' \\
&\quad +\int_{\mathcal{I}_f}\left(\partial_\theta K_C(\theta+\eta,\theta')-\partial_\theta K(\theta,\theta')\right)(u(\theta')-u(\theta+\eta))d\theta' \\
&\equiv I_1+I_2+I_3+I_4
\end{aligned}
$$

where $\mathcal{I}_{s,f}$ were defined in (2.3). We will bound each term individually and eventually recombine. For term I_1 we have, using (2.70),

$$
\begin{aligned}
|I_1| &\leq C\int_{\mathcal{I}_s}|\partial_\theta K_C(\theta,\theta')||u(\theta)-u(\theta')|d\theta' \\
&\leq C\|u\|_{C^{0,\gamma}}\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2}\int_{\mathcal{I}_s}|\theta-\theta'|^{2\gamma-2}d\theta'=C\|u\|_{C^{0,\gamma}}\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2}\eta^{2\gamma-1}.
\end{aligned}
$$

Using the same argument, the same bound holds for term I_2. For term I_3, we may use (2.70) to compute

$$
\begin{aligned}
|I_3| &\leq C\int_{\mathcal{I}_f}|\partial_\theta K_C(\theta,\theta')||u(\theta)-u(\theta+\eta)|d\theta' \\
&\leq C\eta^\gamma\|u\|_{C^{0,\gamma}}\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2}\int_{\mathcal{I}_f}|\theta-\theta'|^{\gamma-2}d\theta'\leq C\|Z\|_{C^{0,\gamma}}\frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^2}{|\boldsymbol{Z}|_*^2}\eta^{2\gamma-1}.
\end{aligned}
$$

For the term I_4, we may use (2.72) to obtain:

$$
\begin{aligned}
|I_4| &\leq C \int_{\mathcal{I}_f} |\triangle_\eta \partial_\theta K_C| \left|u(\theta+h) - u(\theta')\right| d\theta' \\
&\leq C \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^3}{|\boldsymbol{Z}|_*^3} \|u\|_{C^{0,\gamma}} \int_{\mathcal{I}_f} \left(\eta^\gamma \left|\theta - \theta'\right|^{\gamma-2} + \eta \left|\theta - \theta'\right|^{2\gamma-3}\right) \\
&\leq C \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^3}{|\boldsymbol{Z}|_*^3} \|u\|_{C^{0,\gamma}} \eta^{2\gamma-1}.
\end{aligned}
$$

Collecting $I_1, \ldots, I_4$, we see that $\partial_\theta L_C u$ is in $C^{0,2\gamma-1}$ with the desired estimate.

Note that the bound for statement (ii) follows from a simple adaptation of the $\gamma < 1/2$ case. □

Making use of the above proposition, we conclude that:

Proposition 2.3.3. *The operator*

$$
Lu = \int_{\mathbb{S}^1} \frac{\Delta \boldsymbol{Z} \cdot \partial_{\theta'} \boldsymbol{Z}'}{|\Delta \boldsymbol{Z}|^2} u(\theta') d\theta'
$$

maps $L : C^{0,\gamma} \mapsto C^{0,\gamma}$ with bound

$$
\|Lu\|_{C^{0,\gamma}} \leq C \frac{\|\boldsymbol{Z}\|_{C^{1,\gamma}}^3}{|\boldsymbol{Z}|_*^3} \|u\|_{C^{0,\gamma}}
$$

where constant C depends only on γ.

This is obvious given that $\mathcal{H} : C^{0,\gamma} \mapsto C^{0,\gamma}$ and the fact that $L_C : C^{0,\gamma} \mapsto C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor} \subset C^{0,\gamma}$.

It is well understood that $\mathcal{H} : C^{n,\gamma} \mapsto C^{n,\gamma}$ for any $\gamma \in (0,1)$ and $n \geq 0$. We now prove that it is also a bounded operator on $h^{n,\gamma}$ spaces.

Proposition 2.3.4. *For any $n \in \mathbb{N} \cup \{0\}$ and $\gamma \in (0,1)$, the Hilbert transform maps $h^{n,\gamma}$ to $h^{n,\gamma}$.*

Proof. It is well known that for any $u \in C^{n,\gamma}$,

$$
\|\mathcal{H}u\|_{C^{n,\gamma}} \leq C \|u\|_{C^{n,\gamma}} .
$$

It suffices to show that if $u \in h^{0,\gamma}$ then $\mathcal{H}u \in h^{0,\gamma}$. Let $u \in h^{0,\gamma}$. Then, there exists some sequence $\{u_m\}_{n=1}^{\infty}$ such that $u_m \in C^{\infty}$ and $u_m \to u$ in $C^{0,\gamma}$. Since $u_m \in C^{\infty}$, $\mathcal{H}u_m \in C^{\infty}$ as well. Since $\mathcal{H}$ is a bounded operator on $C^{0,\gamma}$ we have

$$\left\|\mathcal{H}u_m - \mathcal{H}u\right\|_{C^{0,\gamma}} \leq C \left\|u_m - u\right\|_{C^{0,\gamma}} .$$

Thus, $\left\|u_m - u\right\|_{C^{0,\gamma}} \to 0$ implies that $\mathcal{H}u$ is the limit of a sequence of smooth functions in $C^{0,\gamma}$ and therefore $\mathcal{H}u \in h^{0,\gamma}$. □

Thus, we also have the following:

Proposition 2.3.5. *The operator*

$$Lu = \int_{\mathbb{S}^1} \frac{\Delta \boldsymbol{Z} \cdot \partial_{\theta'} \boldsymbol{Z}'}{\left|\Delta \boldsymbol{Z}\right|^2} u(\theta') d\theta'$$

maps $L : h^{0,\gamma} \mapsto h^{0,\gamma}$ *with bound*

$$\left\|Lu\right\|_{h^{0,\gamma}} \leq C \frac{\left\|\boldsymbol{Z}\right\|_{C^{1,\gamma}}^3}{\left|\boldsymbol{Z}\right|_*^3} \left\|u\right\|_{h^{0,\gamma}}$$

where constant C *depends only on* γ.

Chapter 3

Generation of an Analytic Semi-group

We wish to write the solution of our systems as

$$\boldsymbol{X}(t) = e^{At}\boldsymbol{X}_0 + \int_0^t e^{A(t-s)}\mathcal{R}(\boldsymbol{X}(s))ds,$$

for some linear operator A. In order to do this, we must show that we can rewrite our equation as

$$\partial_t \boldsymbol{X} = A\boldsymbol{X} + \mathcal{R}(\boldsymbol{X})$$

for some operator A which generates an analytic semigroup. We must do this in both the semilinear and nonlinear setting. In the semilinear setting

$$\begin{aligned}\partial_\theta \boldsymbol{X} &= \frac{1}{4\pi}\int_{\mathbb{S}^1}\left(\partial_{\theta'}\log|\Delta\boldsymbol{X}| - \partial_{\theta'}\left(\frac{\Delta\boldsymbol{X}\otimes\Delta\boldsymbol{X}}{|\Delta\boldsymbol{X}|}\right)\right)\partial_{\theta'}\boldsymbol{X}'d\theta \\ &= -\frac{1}{4\pi}\mathcal{H}\left(\partial_\theta\boldsymbol{X}\right) + \mathcal{R}(\boldsymbol{X})\end{aligned}$$

where

$$\begin{aligned}\mathcal{R}(\boldsymbol{X}) = \frac{1}{4\pi}&\int_{\mathbb{S}^1}\left(\partial_{\theta'}\log|\Delta\boldsymbol{X}| + \frac{1}{2}\cot\left(\frac{\theta-\theta'}{2}\right)\right)\partial_{\theta'}\boldsymbol{X}'d\theta \\ &- \frac{1}{4\pi}\int_{\mathbb{S}^1}\left(\partial_{\theta'}\left(\frac{\Delta\boldsymbol{X}\otimes\Delta\boldsymbol{X}}{|\Delta\boldsymbol{X}|}\right)\right)\partial_{\theta'}\boldsymbol{X}'d\theta.\end{aligned}$$

Thus, in this setting, we need to show that $-\mathcal{H}\partial_\theta$ generates an analytic semigroup.

In the nonlinear setting, we actually do not have to work too much harder. In this case, we need to show that the Frechét derivative of our right hand side generates an analytic semigroup. But as was shown in Section 2.2, the linearization of our right hand side will consist of a sum of integral equations with kernels comprised of functions living in $\mathcal{S}^H_{0,1,\gamma}$ along with a single kernel of the form $\partial_{\theta'} \log|\Delta \boldsymbol{X}|$, which proposition 2.3.2 shows is a lower order perturbation away from the Hilbert Transform. Unlike the semilinear case, there will be mixing of terms because of the linearization of the tension. So, if we can show for some matrix M with component functions in $C^{0,\gamma}$ that $-\mathcal{H}M(\theta)\partial_\theta$ generates an analytic semigroup then $-\mathcal{H}M\partial_\theta + B$ will generate an analytic semigroup for any B which is a lower order operator. So, we will study the operator $-\mathcal{H}M\partial_\theta$ which will encompass both the semilinear and nonlinear cases.

An operator will generate an analytic semigroup if it is sectorial.

Definition 3.0.1. *An operator A is sectorial if there exists constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, $M > 0$ with*

i.) $\rho(A) \supset S_{\theta,\omega} = \{z \in \mathbb{C} : z \neq \omega, |\arg(z-\omega)| < \theta\}$

ii.) $\left\|(z-A)^{-1}\right\|_{L(X)} \leq \frac{C}{|z-\omega|}$, *for all $z \in S_{\theta,\omega}$.*

The first statement says that the spectrum of A lives in a sector and the second statement is the resolvent identity.

3.1 Semigroup Generation in the Constant Matrix Coefficient case

We first show generation of a semigroup in the case where the matrix coefficient is constant. To do this, we will make use of the following theorem which we have adapted from [33].

Theorem 3.1.1 ([33])**.** *If T is a Fourier multiplier operator with multiplier $m \in C^s(\mathbb{R}^n \setminus \{0\}) \cap L^\infty(\mathbb{R}^n)$ for $s > n/2$ and*

$$\left|\partial_\xi^\alpha m(\xi)\right| \leq C_\alpha |\xi|^{-|\alpha|},$$

then for all compactly supported $f \in C^{0,\gamma}$ with $\gamma \in (0,1)$

$$[Tf]_{C^{0,\gamma}} \leq C D_m \, [f]_{C^{0,\gamma}}$$

where the constant C depends only on γ, s, n and $D = \max_{|\alpha| \leq s} C_\alpha$.

This theorem only provides bounds for the seminorm. We will need to augment it to control the full Hölder norm. The following will allow us to do so.

Proposition 3.1.2. *If $[u]_{C^{0,\gamma}} < \infty$ for some $\gamma \in (0,1)$ and $\hat{u}(\xi) \equiv 0$ in a neighborhood of $\xi = 0$, then $\|u\|_0 \leq C\, [u]_{C^{0,\gamma}}$*

Proof. Let u be as stated with $\hat{u}(\xi) \equiv 0$ for $|\xi| \leq \epsilon$, for some $\epsilon > 0$. Let $\varphi(\xi) \in C^\infty(\mathbb{R})$ be compactly supported on $(-\epsilon, \epsilon)$ which is scaled so that $\int_{\mathbb{R}} \check{\varphi}(x)dx = 1$. Since $\check{\varphi} * u \equiv 0$, we have

$$\begin{aligned} |u(x)| = |u(x) - (\check{\varphi} * u)\, x| &= \left| \int_{\mathbb{R}} \check{\varphi}(x-y)(u(x) - u(y))dy \right| \\ &\leq [u]_{C^{0,\gamma}} \int_{\mathbb{R}} |\check{\varphi}(x-y)|\, |x-y|^\gamma \leq C\, [u]_{C^{0,\gamma}}\,, \end{aligned}$$

since φ is smooth. □

Thus, combining these two results yields:

Proposition 3.1.3. *If T is a Fourier multiplier operator as in Theorem 3.1.1, then*

$$\|Tu\|_0 \leq CD_m \|u\|_{C^{0,\gamma}}$$

for all compactly supported $u \in C^{0,\gamma}(\mathbb{R})$.

Proof. Let $u \in C^{0,\gamma}$ have compact support and let $\varphi(\xi)$ be a smooth bump function with compact support such that $\varphi(\xi) \equiv 1$ in a neighborhood of $\xi = 0$. Using this,

$$\|Tu\|_0 = \left\|\mathcal{F}^{-1} m(\xi) \mathcal{F} u\right\|_0 \leq \left\|\mathcal{F}^{-1} m(\xi)(1 - \varphi(\xi))\mathcal{F} u\right\|_0 + \left\|\mathcal{F}^{-1} m(\xi)\varphi(\xi)\mathcal{F} u\right\|_0.$$

Since $\varphi \equiv 1$ near $\xi = 0$, we can apply proposition 3.1.2 to the first term. Thus,

$$\left\|\mathcal{F}^{-1} m(\xi)(1 - \varphi(\xi))\mathcal{F} u\right\|_0 \leq C \left[\mathcal{F}^{-1} m(\xi)(1 - \varphi(\xi))\mathcal{F} u\right]_{C^{0,\gamma}} \leq CD_m \|u\|_{C^{0,\gamma}}\,,$$

where we have also used Theorem 3.1.1 in the last inequality. For the second term, define

$$K(x) = \mathcal{F}^{-1}\left(m(\xi)\varphi(\xi)\right).$$

Then,

$$\left\|\mathcal{F}^{-1}m(\xi)\varphi(\xi)\mathcal{F}u\right\|_0 = \|K * u\|_0 \leq \|K\|_{L^1}\|u\|_0 \leq \|K\|_{L^1}\|u\|_{C^{0,\gamma}}.$$

All that is left is to estimate the L^1 norm of K.

$$\begin{aligned}\|K\|_{L^1} = \int_{\mathbb{R}} |K(x)|\,dx &\leq \left(\int_{\mathbb{R}} |K(x)|^2\left(1+x^2\right)\right)^{1/2}\left(\int_{\mathbb{R}}\left(1+x^2\right)\right)^{1/2} \\ &\leq C\left(\int_{\mathbb{R}} |K(x)|^2\,|1+x|^2\right)^{1/2}.\end{aligned}$$

Using Plancherel's Theorem,

$$\left(\int_{\mathbb{R}} |K(x)|^2\,|1+x|^2\right)^{1/2} = \left(\int_{\mathbb{R}} (m\varphi)^2 + \left(\frac{d}{d\xi}(m\varphi)\right)^2 dx\right)^{1/2} \leq CD_m,$$

for some C which depends only on φ and its support.

□

Finally, using these we are able to prove the following resolvant estimates which when augmented slightly prove generation in the constant coefficient case. Indeed, we may change the bound $\frac{C}{|z|}$ to $\frac{C}{|z-\omega|}$ by changing the constant C if necessary.

Proposition 3.1.4. *Given M, a symmetric positive definite matrix, there exists a sector $S_{\varphi,\omega}$ of the complex plane such that for all $z \in S_{\varphi,\omega}$*

$$\left\|\left(z + M\mathcal{H}\partial_\theta\right)^{-1}\boldsymbol{u}\right\|_{C^{0,\gamma}} \leq \frac{C(1+\lambda_1)}{|z|}\|\boldsymbol{u}\|_{C^{0,\gamma}} \tag{3.1}$$

and

$$\left\|\left(z + M\mathcal{H}\partial_\theta\right)^{-1}\boldsymbol{u}\right\|_{C^{1,\gamma}} \leq C(1+\lambda_0^{-1}+\lambda_1)\|\boldsymbol{u}\|_{C^{0,\gamma}} \tag{3.2}$$

where $0 < \lambda_0 \le \lambda_1$ are the eigenvalues of matrix M and constant C depends only on the sector and γ.

Proof. We wish to prove that

$$\left\|(z + M\mathcal{H}\partial_\theta)^{-1}\,\boldsymbol{u}\right\|_{C^{0,\gamma}} \le \frac{C(1+\lambda_1)}{|z|}\,\|\boldsymbol{u}\|_{C^{0,\gamma}}\,.$$

where λ_1 is the largest eigenvalue of matrix M. However, by assumption, M is a symmetric positive definite matrix and is therefore diagonalizable by an orthogonal matrix. Since this is the case, we have some orthogonal matrix P such that $D = P^{-1}MP$. Thus,

$$z + M\mathcal{H}\partial_\theta = P^{-1}\left(z + D\mathcal{H}\partial_\theta\right)P$$

so,

$$(z + M\mathcal{H}\partial_\theta)^{-1} = P^{-1}\left(z + D\mathcal{H}\partial_\theta\right)^{-1}P.$$

Using this,

$$\begin{aligned}
\left\|(z + M\mathcal{H}\partial_\theta)^{-1}\,\boldsymbol{u}\right\|_{C^{0,\gamma}} &= \left\|P^{-1}\left(z + D\mathcal{H}\partial_\theta\right)^{-1}P\boldsymbol{u}\right\|_{C^{0,\gamma}} = \left\|\left(z + D\mathcal{H}\partial_\theta\right)^{-1}P\boldsymbol{u}\right\|_{C^{0,\gamma}} \\
&\le \left\|\left(z + D\mathcal{H}\partial_\theta\right)^{-1}\right\|\,\|P\boldsymbol{u}\|_{C^{0,\gamma}} \\
&= \left\|\left(z + D\mathcal{H}\partial_\theta\right)^{-1}\right\|\,\|\boldsymbol{u}\|_{C^{0,\gamma}}\,.
\end{aligned}$$

The same holds if we wish to prove statement (3.2). So, it suffices to prove the statement for a diagonal matrix with positive entries. Further, in the case where M is diagonal, we have

$$\begin{aligned}
(z + M\mathcal{H}\partial_\theta)^{-1}\boldsymbol{u} &= \mathcal{F}^{-1}\left(\left((z + M\mathcal{H}\partial_\theta)^{-1}\right)\mathcal{F}\boldsymbol{u}\right) \\
&= \mathcal{F}^{-1}\left(\begin{pmatrix} \frac{1}{z+m_{11}|k|} & 0 \\ 0 & \frac{1}{z+m_{22}|k|} \end{pmatrix}\mathcal{F}\boldsymbol{u}\right).
\end{aligned}$$

But, this operator acts component wise. Thus, it suffices to show that

$$\left\|(z+a\mathcal{H}\partial_\theta)^{-1}u\right\|_{C^{0,\gamma}} \leq \frac{C(a+1)}{|z|}\left\|u\right\|_{C^{0,\gamma}} \tag{3.3}$$

$$\left\|(z+a\mathcal{H}\partial_\theta)^{-1}u\right\|_{C^{1,\gamma}} \leq C(1+a^{-1}+a)\left\|u\right\|_{C^{0,\gamma}} \tag{3.4}$$

for any $a > 0$.

We start with (3.3). It has been shown that the spectrum of this operator lies on the negative real line with an eigenvalue at $z = 0$. We therefore may start by considering any spectrum with generic $\theta \in (\pi/2, \pi)$ and any $\omega > 0$. It is equivalent to get the same bound for a periodic extension on the real line. On the real line, the operator $(z+d\mathcal{H}\partial_\theta)^{-1}(\cdot) = \mathcal{F}^{-1}\left(\frac{1}{z+a|\xi|}\mathcal{F}(\cdot)\right)$. Let $u \in C^{0,\gamma}$ be 2π periodic and let $\{\psi_j\}$ be a partition of unity on the real line where each ψ_j is a copy of ϕ, a smooth bump function with support on $(-2\pi, 2\pi)$, centered at $2\pi j$. Using this,

$$u(x) = \sum_{j=-\infty}^{\infty}\psi_j(x)u(x) = \sum_{j=-\infty}^{\infty}\phi(x-2\pi j)u(x) =: \sum_{j=-\infty}^{\infty}\tilde{u}(x-2\pi j) =: \sum_{j=-\infty}^{\infty}\tilde{u}_j(x).$$

We wish to show that

$$\left\|\mathcal{F}^{-1}\left(\frac{1}{z+a|\xi|}\mathcal{F}u\right)\right\|_{C^{0,\gamma}(-\pi,\pi)} \leq \frac{C(1+a)}{|z|}\left\|u\right\|_{C^{0,\gamma}},$$

for all z in some sector with vertex ω. We have,

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left(\frac{1}{z+a|\xi|}\mathcal{F}u\right)\right\|_{C^{0,\gamma}} = \left\|\mathcal{F}^{-1}\left(\frac{1}{z+a|\xi|}\mathcal{F}\sum_{-\infty}^{\infty}\tilde{u}(x-2\pi j)\right)\right\|_{C^{0,\gamma}} \\
&\leq \sum_{j=-1}^{1}\left\|\mathcal{F}^{-1}\left(\frac{1}{z+a|\xi|}\mathcal{F}\tilde{u}(x-2\pi j)\right)\right\|_{C^{0,\gamma}} + \sum_{|j|\geq 2}\left\|\mathcal{F}^{-1}\left(\frac{1}{z+a|\xi|}\mathcal{F}\tilde{u}(x-2\pi j)\right)\right\|_{C^{0,\gamma}}.
\end{aligned} \tag{3.5}$$

Note that our operator is a Fourier multiplier operator as in Theorem 3.1.1 with $D_m = C(1+a)/|z|$. Using Theorem 3.1.1 and proposition 3.1.3 in combination yields the desired bound for the first of the two above terms as each $\tilde{u}_j$ has compact support. In order to show that the second of the two terms has finite sum with the appropriate bound, we first investigate the decay properties of $\mathcal{F}^{-1}\left(\frac{1}{z+a|\xi|}\right)$. Let

$$K(x) := \mathcal{F}^{-1}\left(\frac{1}{z+a|\xi|}\right)$$

so that $\mathcal{F}^{-1}\left(\frac{1}{z+a|\xi|}\mathcal{F}\tilde{u}(x-2\pi j)\right) = K * \tilde{u}_j$. Integrating by parts twice,

$$\begin{aligned} \left|x^2K(x)\right| &= \left|\int_{\mathbb{R}} x^2 e^{i\xi x}\frac{1}{z+a|\xi|}d\xi\right| = \left|\int_{\mathbb{R}}\left(\frac{d^2}{d\xi^2}e^{i\xi x}\right)\frac{1}{z+a|\xi|}d\xi\right| \\ &\le C\left(\frac{a}{|z|^2} + \left|\int_{\mathbb{R}}\frac{a^2}{(z+a|\xi|)^3}d\xi\right|\right) \le C\frac{a}{|z|^2}. \end{aligned}$$

We have already chosen our sector so that $\omega > 0$ and $\pi/2 < \theta < \pi$ and so the sector contains a ball centered at zero. Thus, $\dfrac{1}{|z|} \le C$, for some C which depends on the sector. We have

$$|K(x)| \le C\frac{a}{|z|x^2}.$$

Now, fix j with $|j| \ge 2$. We will estimate $\|K * \tilde{u}_j\|_{C^{0,\gamma}}$. Since we are only interested in one period of u, let $x \in (-\pi,\pi)$. Then, since $\tilde{u}$ is compactly supported on $(-2\pi, 2\pi)$

$$\begin{aligned} |(K * \tilde{u}_j)(x)| &= \left|\int_{\mathbb{R}} K(x-y)\tilde{u}(y-2\pi j)dy\right| = \left|\int_{\mathbb{R}} K(x-y+2\pi j)\tilde{u}(y)dy\right| \\ &\le C\left\|\tilde{u}\right\|_{C^{0,\gamma}}\int_{-2\pi}^{2\pi}\frac{a}{|z||x-y+2\pi j|^2}dy \le C\frac{a}{\pi^2|z|(2|j|-3)^2}\left\|u\right\|_{C^{0,\gamma}}. \end{aligned} \tag{3.6}$$

Without loss of generality let $h > 0$ be small.

$$\begin{aligned} |(K * \tilde{u}_j)(x+h) - (K * \tilde{u}_j)(x)| &= \left|\int_{\mathbb{R}} K(y+2\pi j)(\tilde{u}(x+h-y) - \tilde{u}(x-y))dy\right| \\ &\le \int_{\mathbb{R}} |K(y-x+2\pi j)||\tilde{u}(y+h) - \tilde{u}(y)|dy \\ &\le C\frac{a}{|z|(2|j|-3)^2}\left\|u\right\|_{C^{0,\gamma}}, \end{aligned}$$

since $\tilde{u}$ has compact support and $x \in (-\pi,\pi)$. Combining these two bounds gives

$$\|K * \tilde{u}_j\|_{C^{0,\gamma}} \le C\frac{a}{|z|(2|j|-3)^2}\left\|u\right\|_{C^{0,\gamma}}.$$

Since $\frac{1}{j^2}$ is summable, we conclude that

$$\sum_{|j|\geq 2} \left\| \mathcal{F}^{-1} \frac{1}{z+a|\xi|} \mathcal{F} \tilde{u}_j \right\|_{C^{0,\gamma}} \leq \frac{Ca}{|z|} \left\| u \right\|_{C^{0,\gamma}} .$$

Plugging this into (3.5) gives

$$\left\| \mathcal{F}^{-1} \frac{1}{z+a|\xi|} \mathcal{F} u \right\|_{C^{0,\gamma}} \leq \frac{C(1+a)}{|z|} \left\| u \right\|_{C^{0,\gamma}} \tag{3.7}$$

for all z in any sector $\omega > 0$ and $\pi/2 < \theta < \pi$.

We now prove bound

$$\left\| (z + a\mathcal{H}\partial_\theta)^{-1} u \right\|_{C^{1,\gamma}} \leq C \left\| u \right\|_{C^{0,\gamma}} .$$

Using the sector and bound from before, we already have the bound

$$\left\| (z + a\mathcal{H}\partial_\theta)^{-1} u \right\|_0 \leq \frac{C(a+1)}{|z|} \left\| u \right\|_{C^{0,\gamma}} \leq C(a+1) \left\| u \right\|_{C^{0,\gamma}}$$

as $|z|$ is bounded away from zero by virtue of being in the chosen sector. Thus, we need only find

$$\left\| \partial_\theta (z + a\mathcal{H}\partial_\theta)^{-1} u \right\|_{C^{0,\gamma}} \leq C(a) \left\| u \right\|_{C^{0,\gamma}} ,$$

for some constant C which will depend on a in some way. Thus, it suffices to study the operator

$$\mathcal{F}^{-1} \left(\frac{\xi}{z+a|\xi|} \mathcal{F}(\cdot) \right) .$$

We proceed as we did for the previous bound by using the same partition of unity $\{\psi_j\}$ and the bound

$$\begin{aligned} \left\| \mathcal{F}^{-1} \left(\frac{\xi}{z+a|\xi|} \mathcal{F} u \right) \right\|_{C^{0,\gamma}} &= \left\| \mathcal{F}^{-1} \left(\frac{\xi}{z+a|\xi|} \mathcal{F} \sum_{-\infty}^{\infty} \tilde{u}(x-2\pi j) \right) \right\|_{C^{0,\gamma}} \\ \leq \sum_{j=-1}^{1} \left\| \mathcal{F}^{-1} \left(\frac{\xi}{z+a|\xi|} \mathcal{F} \tilde{u}(x-2\pi j) \right) \right\|_{C^{0,\gamma}} &+ \sum_{|j|\geq 2} \left\| \mathcal{F}^{-1} \left(\frac{\xi}{z+a|\xi|} \mathcal{F} \tilde{u}(x-2\pi j) \right) \right\|_{C^{0,\gamma}} . \end{aligned}$$

As before, we can use Theorem 3.1.1 and proposition 3.1.2 to control the first set of terms. Note that for this new operator, $D_m = C(1+a^{-1})$ for some constant C which depends on the sector. We now estimate the decay properties of kernel $K(x) := \mathcal{F}^{-1}\left(\frac{\xi}{z+a|\xi|}\right)$. As before

$$x^2 K(x) = -\int_{\mathbb{R}} \frac{d^2}{d\xi^2} e^{ix\xi} \frac{\xi}{z+a|\xi|} d\xi.$$

We have to be careful with the integration by parts. Let us define $\phi_\epsilon(\xi) := \psi(\epsilon\xi)$ such that ψ is a smooth function with compact support so that $\lim_{\epsilon\to 0} \phi_\epsilon(\xi) = 1$ everywhere. Using this,

$$x^2 K(x) = -\lim_{\epsilon\to 0} \int_{\mathbb{R}} \frac{d^2}{d\xi^2} \left(e^{ix\xi}\right) \frac{\xi\phi_\epsilon(\xi)}{z+a|\xi|} d\xi$$

in the distributional sense. Using the compact support of ϕ_ϵ, we may integrate by parts twice and find

$$\begin{aligned} x^2 K(x) &= -\lim_{\epsilon\to 0} \int_{\mathbb{R}} e^{ix\xi} \frac{d^2}{d\xi^2} \left(\frac{\xi\phi_\epsilon(\xi)}{z+a|\xi|}\right) d\xi \\ &= -\lim_{\epsilon\to 0} \int_{\mathbb{R}} e^{ix\xi} \left(\frac{-2az\phi_\epsilon(\xi)}{(z+a|\xi|)^3} + \frac{z\phi_\epsilon'(\xi)}{(z+a|\xi|)^2} + \frac{\xi\phi_\epsilon''(\xi)}{z+a|\xi|}\right) d\xi. \end{aligned}$$

The first of these terms is clearly bounded by an integrable function and so using the Dominated Convergence Theorem yields

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}} e^{ix\xi} \frac{-2az\phi_\epsilon(\xi)}{(z+a|\xi|)^3} d\xi = \int_{\mathbb{R}} e^{ix\xi} \frac{-2az}{(z+a|\xi|)^3} d\xi$$

which has bound

$$\left|\int_{\mathbb{R}} e^{ix\xi} \frac{-2az}{(z+a|\xi|)^3} d\xi\right| \le 4\int_0^\infty \frac{a|z|}{|z+a\xi|^3} d\xi \le \frac{C}{|z|}. \tag{3.8}$$

We will now show that the remaining two terms both limit to zero. For the second term, note that $\phi_\epsilon'(\xi) = \epsilon\psi'(\epsilon\xi)$ is uniformly bounded by $\epsilon \|\psi\|_{C^\infty}$. Because

$$\int_{\mathbb{R}} \frac{|z|}{|z+a|\xi||^2} d\xi \le \frac{C}{a},$$

we have

$$\lim_{\epsilon\to 0}\left|\int_{\mathbb{R}} e^{ix\xi}\frac{z\phi_\epsilon'(\xi)}{(z+a|\xi|)^2}d\xi\right| \leq \lim_{\epsilon\to 0}\frac{C\epsilon}{a}\int_{\mathbb{R}}\frac{|z|}{|z+a|\xi||^2}d\xi = 0.$$

Using the change of variables $\tilde{\xi} = \epsilon\xi$ and dropping the tildes, we can rewrite the third term as

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{\mathbb{R}}\epsilon e^{ix\xi/\epsilon}\left(\frac{\xi}{z+a|\xi|}\right)\psi''(\xi)d\xi\\ &=\lim_{\epsilon\to 0}\int_{\mathbb{R}}\epsilon e^{ix\xi/\epsilon}\left(\frac{\xi}{a|\xi|}-\frac{\epsilon z\xi}{a(z+a|\xi|)|\xi|}\right)\psi''(\xi)d\xi.\end{aligned}$$

Since ψ has compact support and is flat near $\xi = 0$, we can rewrite this as

$$\lim_{\epsilon\to 0}\int_{\xi_1\leq|\xi|\leq\xi_2}\epsilon e^{ix\xi/\epsilon}\left(\frac{\xi}{a|\xi|}-\frac{\epsilon z\xi}{a(z+a|\xi|)|\xi|}\right)\psi''(\xi)d\xi$$

for some $\xi_1, \xi_2 > 0$. The first of these two terms is clearly bounded and integrable. Due to z being bounded away from the negative real axis, for small ϵ, the second term has the bound

$$\left|\frac{\epsilon z\xi}{a(z+a|\xi|)|\xi|}\right| \leq \frac{1}{a}\frac{|z|}{|z+\frac{a\xi}{\epsilon}|} \leq C$$

for some constant C which depends on a as well as the sector. Thus,

$$\begin{aligned}&\lim_{\epsilon\to 0}\left|\int_{\xi_1\leq|\xi|\leq\xi_2}\epsilon e^{ix\xi/\epsilon}\left(\frac{\xi}{a|\xi|}-\frac{\epsilon z\xi}{a(z+a|\xi|)|\xi|}\right)\psi''(\xi)d\xi\right|\\ &\leq \lim_{\epsilon\to 0}\epsilon C\|\psi\|_{C^\infty}\int_{\xi_1\leq|\xi|\leq\xi_2}d\xi \leq \lim_{\epsilon\to 0}\epsilon C\|\psi\|_{C^\infty} = 0.\end{aligned}$$

Finally, using the bound found in (3.8)

$$|x^2K(x)| \leq \frac{C}{|z|} \leq C$$

for some C which depends only on the sector. Hence,

$$|K(x)| \leq \frac{C}{|x|^2}.$$

Using this, we may proceed as in (3.6) and find

$$\left\| \partial_\theta (z + a\mathcal{H})^{-1} u \right\|_{C^{0,\gamma}} \leq C(1 + a^{-1}) \|u\|_{C^{0,\gamma}},$$

for some C depends only on the sector. Combining this with the supremum bound gives

$$\left\| (z + a\mathcal{H})^{-1} u \right\|_{C^{1,\gamma}} \leq C(1 + a + a^{-1}) \|u\|_{C^{0,\gamma}}.$$

□

3.1.1 Additional Semigroup Estimates

In the semilinear case, the operator $\Lambda u := -\frac{1}{4}\mathcal{H}(\partial_\theta u)$ is the principal linear part of the evolution. Setting M to be the identity matrix, we have just shown that Λ generates an analytic semi group on the $C^{0,\gamma}$ spaces. In this simple case, we can actually calculate the exact form the semigroup takes and prove sharper estimates than abstract theory would provide which in turn will allow us to prove stronger results.

Given that Λ is just the square root of the one-dimensional Laplacian and is also the Dirichlet-to-Neumann operator on a disk (up to a multiplicative constant), the results we prove in this section are likely well-known to students of this operator.

Suppose we are given a continuous function u on the unit circle $\mathbb{S}^1$. We may express u in terms of Fourier series:

$$u(\theta) = \sum_{k=-\infty}^{\infty} u_k e^{ik\theta},\ u_k = \frac{1}{2\pi}\int_0^{2\pi} u(\theta')e^{-ik\theta'}d\theta'.$$

It is well-known that the Fourier symbol of the operator $\Lambda u = -\frac{1}{4}\mathcal{H}(\partial_\theta u)$ is given by $-|k|/4$ where k is the Fourier wave number. That is to say,

$$\Lambda u = -\frac{1}{4}\sum_{k=-\infty}^{\infty} |k| u_k e^{ik\theta},$$

for sufficiently smooth u. The operator $e^{t\Lambda}, t > 0$ is therefore:

$$e^{t\Lambda}u = \sum_{k=-\infty}^{\infty} u_k e^{ik\theta - |k|t/4}.$$

For $t > 0$, we have:

$$\begin{aligned} e^{t\Lambda}u &= \frac{1}{2\pi}\int_{\mathbb{S}^1} \sum_{k=-\infty}^{\infty} e^{-|k|t/4} e^{ik(\theta-\theta')} u(\theta') d\theta' \\ &= \frac{1}{2\pi}\int_{\mathbb{S}^1} P(e^{-t/4}, \theta - \theta') u(\theta') d\theta', \end{aligned} \tag{3.9}$$

where

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}.$$

Note that $P(r, \theta)$ is the Poisson kernel.

Another useful way to view Λ is as the Dirichlet-to-Neumann map on the unit disk. Consider the following Laplace boundary value problem:

$$\Delta v = 0, \text{ for } \mathbb{D}^2, \; v = u(\theta) \text{ on } \partial\mathbb{D} = \mathbb{S}^1,$$

where $\mathbb{D}^2$ is the open unit disk and θ is the angular coordinate. Define the Dirichlet-to-Neumann map Λ_{DN} as:

$$\Lambda_{\text{DN}} : u \mapsto \left.\frac{\partial v}{\partial r}\right|_{r=1}$$

where r is the radial coordinate. Then $\Lambda = -\Lambda_{\text{DN}}/4$. This explains the appearance of the Poisson kernel in the expression of $e^{t\Lambda}$ above.

We first state a result on the mapping properties of Λ on Hölder spaces.

Proposition 3.1.5. *The operator Λ is a bounded operator from $C^{k+1,\gamma}(\mathbb{S}^1)$ to $C^{k,\gamma}(\mathbb{S}^1)$ for $k = 0, 1, 2, \ldots$ and $0 < \gamma < 1$.*

Proof. This is a consequence of the well-known fact that the Hilbert transform is a bounded map from $C^{k,\gamma}(\mathbb{S}^1)$ to itself. □

We turn our attention to the semigroup $e^{t\Lambda}$.

Proposition 3.1.6.

For $u \in C^k(\mathbb{S}^1),\ k \in \{0\} \cup \mathbb{N}$,

$$\left\|e^{t\Lambda}u\right\|_{C^k} \leq \|u\|_{C^k}\text{ , for } 0 < t, \tag{3.10}$$

$$\left\|e^{t\Lambda}u\right\|_{C^{k+1}} \leq \frac{C}{t}\|u\|_{C^k} \text{ for } 0 < t < 1, \tag{3.11}$$

where C above is a constant that does not depend on u (or k).

Proof. Let $r = e^{-t/4}$ in what follows. Inequality (3.10) for $k = 0$ is a simple consequence of the well-known fact that

$$\frac{1}{2\pi}\int_0^{2\pi} P(r, \theta')d\theta' = 1,\ P(r, \theta) > 0.$$

Indeed,

$$\left|(e^{t\Lambda}u)(\theta)\right| \leq \frac{1}{2\pi}\int_0^{2\pi}\left|P(r, \theta - \theta')\right|\left|u(\theta')\right|d\theta' \leq \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta)d\theta\,\|u\|_{C^0} = \|u\|_{C^0}\,.$$

What we proved above is just the maximum principle for harmonic functions on a disk. For $k > 0$, note that

$$\partial_\theta^k\left(e^{t\Lambda}u\right) = e^{t\Lambda}\left(\partial_\theta^k u\right). \tag{3.12}$$

Thus,

$$\left[e^{t\Lambda}u\right]_{C^k} = \left\|e^{t\Lambda}\left(\partial_\theta^k u\right)\right\|_{C^0} \leq \left\|\partial_\theta^k u\right\|_{C^0} = [u]_{C^k}\,, \tag{3.13}$$

where inequality (3.10) for $k = 0$ was used in the inequality above. This concludes the proof of (3.10).

Consider (3.11) for $k = 0$.

$$\left|\partial_\theta(e^{t\Lambda}u)\right| = \left|\frac{1}{2\pi}\int_0^{2\pi}\partial_\theta P(r, \theta - \theta')u(\theta')d\theta'\right| \leq \frac{1}{\pi}\int_0^{\pi}\left|\partial_\theta P(r, \theta)\right|d\theta\,\|u\|_{C^0}\,,$$

where we used the symmetry of P with respect to θ in the last inequality. Note that $P(r, \theta)$ is a decreasing function of θ from 0 to π. Thus,

$$\int_0^{\pi}\left|\partial_\theta P(r, \theta)\right|d\theta = -\int_0^{\pi}\partial_\theta P(r, \theta)d\theta = P(r, 0) - P(r, \pi) = \frac{2(1 + r^2)}{1 - r^2}.$$

Thus, we have:

$$\left[e^{t\Lambda}u\right]_{C^1} \leq \frac{2(1 + e^{-t/2})}{\pi(1 - e^{-t/2})}\|u\|_{C^0} \leq \frac{2}{\pi}\left(1 + \frac{4}{t}\right)\|u\|_{C^0}\,.$$

Using (3.12) and proceeding as in (3.13), we find:

$$\left[e^{t\Lambda}u\right]_{C^{k+1}} \leq \frac{2}{\pi}\left(1+\frac{4}{t}\right)[u]_{C^k}\,.$$

Inequality (3.11) now follows with $C = 1 + 10/\pi$. □

To state the next proposition, we introduce the following notation. For $\alpha > 0, \alpha \notin \mathbb{N}$, we let $C^{\alpha}(\mathbb{S}^1) = C^{\lfloor\alpha\rfloor,\alpha-\lfloor\alpha\rfloor}(\mathbb{S}^1)$ where $\lfloor\alpha\rfloor$ is the largest integer smaller than α. This next proposition will be key for proving the local existence of mild solutions.

Proposition 3.1.7 (Hölder estimates on Semigroup)**.** *Let $u \in C^{\alpha}(\mathbb{S}^1), \alpha \geq 0$ and let $\beta \geq 0$ satisfy $0 \leq \beta - \alpha \leq 1$. Then,*

$$\left\|e^{t\Lambda}u\right\|_{C^\beta} \leq \frac{C}{t^{\beta-\alpha}}\left\|u\right\|_{C^\alpha},\ 0 < t < 1, \tag{3.14}$$

where the constant C above depends only on α and β.

To prove the above proposition, we shall make use of the following standard interpolation result which can be found, for example, in Chapter 1 of [21].

Proposition 3.1.8 (Interpolation of Bounded Operators on Hölder Spaces)**.** *Let $0 \leq \alpha_0 \leq \alpha_1$ and $0 \leq \beta_0 \leq \beta_1$ and let $\mathcal{T}$ be a bounded operator from $C^{\alpha_i}(\mathbb{S}^1)$ to $C^{\beta_i}(\mathbb{S}^1), i = 0, 1$ so that:*

$$\left\|\mathcal{T}u\right\|_{C^{\beta_i}} \leq K_i\left\|u\right\|_{C^{\alpha_i}},\ i = 0, 1,$$

where $u \in C^{\alpha_i}(\mathbb{S}^1)$ and the constant $K_i > 0$ does not depend on u. Let $0 < \sigma < 1$ and $\alpha_\sigma = (1-\sigma)\alpha_0 + \sigma\alpha_1,\ \beta_\sigma = (1-\sigma)\beta_0 + \sigma\beta_1$. Suppose one of the following conditions is satisfied.

(i) $\alpha_\sigma \notin \mathbb{N}$ *and* $\beta_\sigma \notin \mathbb{N}$.

(ii) $\alpha_0 = \alpha_1$ *and* $\beta_\sigma \notin \mathbb{N}$.

(iii) $\alpha_\sigma \notin \mathbb{N}$ *and* $\beta_0 = \beta_1$.

Then, $\mathcal{T}$ defines a bounded operator from $C^{\alpha_\sigma}(\mathbb{S}^1)$ to $C^{\beta_\sigma}(\mathbb{S}^1)$ so that:

$$\left\|\mathcal{T}u\right\|_{C^{\beta_\sigma}} \leq CK_0^{1-\sigma}K_1^{\sigma}\left\|u\right\|_{C^{\alpha_\sigma}}$$

where $u \in C^{\alpha_\sigma}(\mathbb{S}^1)$ and the constant $C > 0$ does not depend on $\mathcal{T}$ (or u) and depends only on $\alpha_i, \beta_i, i = 0, 1$ and σ.

Remark 3.1.9. *The restriction to non-integer values in the above proposition can be lifted if we replace the definition of C^k spaces for integer k with Hölder-Zygmund spaces. We do not need such results in this paper. See [21] for details.*

Proof of Proposition 3.1.7. Consider (3.10) at two adjacent integer levels:

$$\left\|e^{t\Lambda}u\right\|_{C^k} \leq \|u\|_{C^k}\,,\ \left\|e^{t\Lambda}u\right\|_{C^{k+1}} \leq \|u\|_{C^{k+1}}\,.$$

Interpolating between these two inequalities using Proposition 3.1.8, we obtain

$$\left\|e^{t\Lambda}u\right\|_{C^\alpha} \leq C\,\|u\|_{C^\alpha} \tag{3.15}$$

for $k < \alpha < k+1$, where C depends only on α. Since k was arbitrary, this establishes (3.14) for $\beta = \alpha$. Likewise, considering (3.11) at adjacent integer values, we obtain:

$$\left\|e^{t\Lambda}u\right\|_{C^{\gamma+1}} \leq \frac{C}{t}\,\|u\|_{C^\gamma} \tag{3.16}$$

for any $\gamma \geq 0$. This establishes (3.14) for $\beta = \alpha + 1$. Interpolating between (3.15) and (3.16) setting $\alpha = \gamma$, we obtain the rest of the cases of (3.14) so long as β is not an integer. To handle the case when β is an integer, interpolate (3.15) and (3.16) setting $\alpha = \gamma + 1$. □

Remark 3.1.10. *The estimates in proposition* (3.1.7) *can also be derived by using the fact that* Λ *generates an analytic semigroup on* $C^{0,\gamma}$ *by making the appropriate choice of domain* $D(\Lambda)$.

Finally, we turn to strong continuity of the the semigroup operator $e^{t\Lambda}$. To this end, recall the little Hölder spaces $h^{k,\gamma}(\mathbb{S}^1), k = 0,1,2,\cdots, 0 < \gamma < 1$ as the completion of $C^\infty(\mathbb{S}^1)$ (smooth functions on the unit circle) in $C^{k,\gamma}(\mathbb{S}^1)$. A function $u \in C^{k,\gamma}(\mathbb{S}^1)$ belongs to $h^{k,\gamma}(\mathbb{S}^1)$ if and only if:

$$\lim_{\delta\searrow 0}\sup_{|\theta-\theta'|<\delta}\frac{\left|u^{(k)}(\theta) - u^{(k)}(\theta')\right|}{|\theta-\theta'|^\gamma} = 0,$$

where $u^{(k)}$ is the k-th derivative of u. From this, it is immediate that $C^{k,\beta}(\mathbb{S}^1)$ embeds continuously into $h^{k,\alpha}(\mathbb{S}^1)$ for any $\beta > \alpha$. We refer to Chapter 0 of [21] for a discussion of these issues.

Given that the the space of smooth functions is not dense in $C^{k,\gamma}(\mathbb{S}^1)$, we can only expect strong continuity of $e^{t\Lambda}$ in the little Hölder space $h^{k,\gamma}(\mathbb{S}^1)$.

Proposition 3.1.11. *Let $u \in C^{k,\gamma}(\mathbb{S}^1),\ k = 0, 1, 2, \ldots$ and $0 < \gamma < 1$. Then,*

$$\lim_{t \searrow 0} \left\| e^{t\Lambda} u - u \right\|_{C^{k,\gamma}} = 0 \tag{3.17}$$

if and only if $u \in h^{k,\gamma}(\mathbb{S}^1)$.

Proof. Given that $e^{t\Lambda} u$ is a smooth function for $t > 0$, (3.17) can only be true if $u \in h^{k,\gamma}(\mathbb{S}^1)$. To prove the converse, suppose $u \in h^{k,\gamma}(\mathbb{S}^1)$. Then, by definition, for any $\epsilon > 0$, there is a function $u_\epsilon \in C^\infty(\mathbb{S}^1)$ such that $\|u - u_\epsilon\|_{C^{k,\gamma}} \leq \epsilon$. By the well-known properties of the Poisson kernel,

$$\lim_{t \searrow 0} \left\| e^{t\Lambda} u_\epsilon - u_\epsilon \right\|_{C^{k+1}} = 0.$$

Since the C^{k+1} norm dominates the $C^{k,\gamma}$ norm, the above is true in the $C^{k,\gamma}$ norm. The desired result now follows from the boundedness of $e^{t\Lambda}$ as a bounded operator from $C^{k,\gamma}(\mathbb{S}^1)$ to itself, as shown in (3.14) (or (3.15)). □

3.2 Semigroup Generation with Variable Coefficients

We now move onto the case where matrix M is allowed to vary in θ. In addition to proposition 3.1.4, we will need a few extra tools to prove generation in this case. In particular, we will need the following commutator estimate on the Hilbert Transform.

Lemma 3.2.1. *Given $\phi \in C^\infty$ and $\psi \in C^0$, for any $0 < \alpha < 1$*

$$\left\| [\mathcal{H}, \phi]\, \psi \right\|_{C^{0,\alpha}} \leq C \left\| \phi \right\|_{C^\infty} \left\| \psi \right\|_0$$

Proof. Let $\phi \in C^\infty$ and $\psi \in C^0$. We may rewrite the operation $[\mathcal{H}, \phi]\psi$ as

$$\begin{aligned}
[\mathcal{H}, \phi]\psi(\theta) &= \frac{1}{2\pi} \int_{\mathbb{S}^1} \cot\left(\frac{\theta - \theta'}{2}\right) \big(\phi(\theta') - \phi(\theta)\big)\, \psi(\theta') d\theta' \\
&= \frac{1}{2\pi} \int_{\mathbb{S}^1} \left(\cot\left(\frac{\theta - \theta'}{2}\right) - \frac{2}{\theta - \theta'}\right) \big(\phi(\theta') - \phi(\theta)\big)\, \psi(\theta') d\theta' \\
&- \frac{1}{\pi} \int_{\mathbb{S}^1} \frac{\phi(\theta') - \phi(\theta)}{\theta' - \theta} \psi(\theta') d\theta'.
\end{aligned}$$

Making use of the series expansion of $\cot(x)$ and the triangle inequality,

$$|[\mathcal{H},\phi]\psi(\theta)| \leq C \left\|\phi\right\|_{C^\infty} \left\|\psi\right\|_{C^0},$$

for some constant C which depends only on the expansion of $\cot(x)$. Now, without loss of generality, let $\eta > 0$. We need to estimate

$$\begin{aligned}
[\mathcal{H},\phi]\psi(\theta+\eta) - [\mathcal{H},\phi]\psi(\theta) &= \frac{1}{2\pi}\int_{\mathbb{S}^1} \cot\left(\frac{\theta+\eta-\theta'}{2}\right)\big(\phi(\theta')-\phi(\theta+\eta)\big)\,\psi(\theta')d\theta' \\
&\quad - \frac{1}{2\pi}\int_{\mathbb{S}^1} \cot\left(\frac{\theta-\theta'}{2}\right)\big(\phi(\theta')-\phi(\theta)\big)\,\psi(\theta')d\theta',
\end{aligned}$$

which we can rewrite as

$$\begin{aligned}
[\mathcal{H},\phi]\psi(\theta+\eta) - [\mathcal{H},\phi]\psi(\theta) &= \frac{1}{\pi}\int_{\mathbb{S}^1} \left(\frac{\phi(\theta')-\phi(\theta+\eta)}{\theta+\eta-\theta'} - \frac{\phi(\theta')-\phi(\theta)}{\theta-\theta'}\right)\psi(\theta')d\theta' \\
&\quad + \frac{1}{2\pi}\int_{\mathbb{S}^1} \left(\cot\left(\frac{\theta+\eta-\theta'}{2}\right) - \frac{2}{\theta+\eta-\theta'}\right)\big(\phi(\theta')-\phi(\theta+\eta)\big)\,\psi(\theta')d\theta' \\
&\quad - \frac{1}{2\pi}\int_{\mathbb{S}^1} \left(\cot\left(\frac{\theta-\theta'}{2}\right) - \frac{2}{\theta-\theta'}\right)\big(\phi(\theta')-\phi(\theta+\eta)\big)\,\psi(\theta')d\theta'.
\end{aligned}$$

Expanding both $\phi(\theta+\eta)$ and $\phi(\theta)$ around θ' yields

$$\left|\frac{\phi(\theta')-\phi(\theta+\eta)}{\theta+\eta-\theta'} - \frac{\phi(\theta')-\phi(\theta)}{\theta-\theta'}\right| \leq C\eta \left\|\phi\right\|_{C^\infty}.$$

Thus,

$$\left|\frac{1}{\pi}\int_{\mathbb{S}^1} \left(\frac{\phi(\theta')-\phi(\theta+\eta)}{\theta+\eta-\theta'} - \frac{\phi(\theta')-\phi(\theta)}{\theta-\theta'}\right)\psi(\theta')d\theta'\right| \leq C\eta \left\|\phi\right\|_{C^\infty} \left\|\psi\right\|_{C^0}.$$

Similarly, using the expansion of $\cot x$ we find

$$\left|\left(\cot\left(\frac{\theta+\eta-\theta'}{2}\right) - \frac{2}{\theta+\eta-\theta'}\right) - \left(\cot\left(\frac{\theta-\theta'}{2}\right) - \frac{2}{\theta-\theta'}\right)\right| \leq C\eta,$$

which implies the difference between the second to terms above is bounded by $C\eta \left\|\phi\right\|_{C^\infty} \left\|\psi\right\|_{C^0}$ as well. Together, we have

$$|[\mathcal{H},\phi]\psi(\theta+\eta)-[\mathcal{H},\phi]\psi(\theta)|\leq C\eta\left\|\phi\right\|_{C^\infty}\left\|\psi\right\|_{C^0},$$

and so the operator $[\mathcal{H},\phi]$ is Lipschitz continuous and therefore in $C^{0,\alpha}$ for any $\alpha\in(0,1)$.

□

For simplicity, let us define the operator Λ_M as

$$\Lambda_M u=-\mathcal{H}\left(M(\theta)\partial_\theta u\right), \tag{3.18}$$

for a given matrix $M(\theta)$ with component functions $m_{ij}\in C^{0,\gamma}(\mathbb{S}^1)$.

Proposition 3.2.2. *Let Λ_M be defined as above for some symmetric positive definite matrix $M(\theta)$ with component functions in $C^{1,\gamma}$. Further, let $B : C^{0,\gamma}\mapsto C^\beta$ for some $\beta>\gamma$ with $C^\beta=C^{\lfloor\beta\rfloor,\beta-\lfloor\beta\rfloor}$ and $\beta\notin\mathbb{N}$. Then, there exists some sector $S_{\varphi,\omega}$ such that for all z in the sector.*

$$\begin{aligned}\left\|(z-(\Lambda_M+B))\boldsymbol{u}\right\|_{C^{0,\gamma}}&\geq\frac{C}{1+\lambda_1}|z|\left\|\boldsymbol{u}\right\|_{C^{0,\gamma}}\\ \left\|(z-(\Lambda_M+B))\boldsymbol{u}\right\|_{C^{0,\gamma}}&\geq C\frac{1}{1+\lambda_1+\lambda_0^{-1}}\left\|\boldsymbol{u}\right\|_{C^{1,\gamma}}\end{aligned}$$

where λ_1 is the largest eigenvalue of matrix $M(\theta)$ over any $\theta\in\mathbb{S}^1$ and λ_0 is the smallest and C depends only on the choice of sector, γ and the operator norm of B.

Proof. We make the initial choice of taking $z\in S_{\varphi,\omega}$, where sector $S_{\varphi,\omega}$ is chosen as that of proposition 3.1.4. Let us define

$$A=\Lambda_M+B. \tag{3.19}$$

Let $\boldsymbol{u}\in C^{1,\gamma}$ and let $\{\phi_j\}_{j=1}^n$ be a partition of unity on $\mathbb{S}^1$ with ϕ_j centered around points θ_j. Then,

$$\boldsymbol{u}=\sum_{j=1}^n\phi_j\boldsymbol{u}\equiv\sum_{j=1}^n\boldsymbol{u}_j.$$

Define $M_j \equiv M(\theta_j)$. Also, let λ_1 be the largest eigenvalue of matrix $M(\theta)$ over any $\theta \in \mathbb{S}^1$ and let λ_0 be the smallest. Using these, we rewrite

$$(z - \Lambda_M)\boldsymbol{u} = z\boldsymbol{u} + \mathcal{H}(M(\theta)\partial_\theta \boldsymbol{u}) = z\boldsymbol{u} + M_j \mathcal{H}\left(\partial_\theta \boldsymbol{u}\right) + \mathcal{H}\left(\left(M(\theta) - M_j\right)\partial_\theta \boldsymbol{u}\right).$$

For a single $\boldsymbol{u}_j$,

$$\begin{aligned}
\left\|(z - \Lambda_M)\boldsymbol{u}_j\right\|_{C^{0,\gamma}} &\geq \left\|(z + M_j\mathcal{H}\partial_\theta)\boldsymbol{u}_j\right\|_{C^{0,\gamma}} - \left\|\mathcal{H}\left(\left(M - M_j\right)\partial_\theta \boldsymbol{u}_j\right)\right\|_{C^{0,\gamma}} \\
&\geq \left\|(z + M_j\mathcal{H}\partial_\theta)\boldsymbol{u}_j\right\|_{C^{0,\gamma}} - C_1\left\|\left(M - M_j\right)\partial_\theta \boldsymbol{u}_j\right\|_{C^{0,\gamma}} \\
&\geq \left\|(z + M_j\mathcal{H}\partial_\theta)\boldsymbol{u}_j\right\|_{C^{0,\gamma}} - C_1\left\|M - M_j\right\|_0 \left\|\partial_\theta \boldsymbol{u}_j\right\|_{C^{0,\gamma}} \\
&- C_1\left\|M - M_j\right\|_{C^{0,\gamma}} \left\|\partial_\theta \boldsymbol{u}_j\right\|_0,
\end{aligned}$$

for some constant C_1. Note here that we need only evaluate $\left\|M - M_j\right\|_0$ on the support of $\boldsymbol{u}_j$.

Inequalities (3.1) and (3.2) can be interpolated to give a new bound on an intermediate Hölder space. We find,

$$\left\|(z + M\mathcal{H}\partial_\theta)\boldsymbol{u}\right\|_{C^{0,\gamma}} \geq C\kappa(\sigma)\left|z\right|^{1-\sigma}\left\|\boldsymbol{u}\right\|_{C^{\gamma+\sigma}}, \tag{3.20}$$

where $\sigma \in [0,1]$, $C^{\gamma+\sigma} = C^{\lfloor\gamma+\sigma\rfloor,\gamma+\sigma-\lfloor\gamma+\sigma\rfloor}$ as usual, and

$$\kappa(\sigma) = \left(\frac{1}{1 + \lambda_1 + \lambda_0^{-1}}\right)^{\sigma}\left(\frac{1}{1 + \lambda_1}\right)^{1-\sigma}.$$

Using this,

$$\begin{aligned}
\left\|(z + M_j\mathcal{H}\partial_\theta)\boldsymbol{u}_j\right\|_{C^{0,\gamma}} \geq C\frac{1}{1 + \lambda_1}|z|\left\|\boldsymbol{u}_j\right\|_{C^{0,\gamma}} + C\kappa(\sigma)|z|^{1-\sigma}\left\|\boldsymbol{u}_j\right\|_{C^{\gamma+\sigma}} \\
+ C\frac{1}{1 + \lambda_1 + \lambda_0^{-1}}\left\|\boldsymbol{u}_j\right\|_{C^{1,\gamma}}.
\end{aligned}$$

Using $\left\|\partial_\theta \boldsymbol{u}_j\right\|_0 \leq \left\|\boldsymbol{u}_j\right\|_{C^{1,\alpha}}$ for any $0 < \alpha$, we may choose $0 < \sigma < 1$ so

$$\begin{aligned}\|(z+\mathcal{H}M\partial_\theta)\boldsymbol{u}_j\|_{C^{0,\gamma}} \geq{}& \frac{C}{1+\lambda_1}|z|\,\|\boldsymbol{u}_j\|_{C^{0,\gamma}} + \frac{C}{1+\lambda_1+\lambda_0^{-1}}\,\|\boldsymbol{u}\|_{C^{1,\gamma}}\\ &+ C\kappa(\sigma)|z|^{1-\sigma}\,\|\boldsymbol{u}_j\|_{C^{\gamma+\sigma}} - C_1\,\|M-M_j\|_{C^{0,\gamma}}\,\|\boldsymbol{u}_j\|_{C^{\gamma+\sigma}}\\ &- C_1\,\|M-M_j\|_0\,\|\partial_\theta\boldsymbol{u}_j\|_{C^{0,\gamma}}\,.\end{aligned}$$

We may move our sector to some new $S_{\varphi,\omega}$ if needed to make $|z|^{1-\sigma}$ large implying

$$\left(\frac{C}{2}\kappa(\sigma)|z|^{1-\sigma} - C_1\,\|M-M_j\|_{C^{0,\gamma}}\right)\|\boldsymbol{u}_j\|_{C^{\gamma+\sigma}} > 0.$$

Also, we are only interested in $\|M-M_j\|_0$ over the support of ϕ_j, and so, we may change our partition of unity so that

$$\left(\frac{C}{2(1+\lambda_1+\lambda_0^{-1})} - C_1\,\|M-M_j\|_0\right)\|\boldsymbol{u}_j\|_{C^{1,\gamma}} > 0, \tag{3.21}$$

allowing us to discard these terms in the above inequality and conclude

$$\begin{aligned}\|(z+\mathcal{H}M\partial_\theta)\boldsymbol{u}_j\|_{C^{0,\gamma}} \geq{}& C\frac{1}{1+\lambda_1}|z|\,\|\boldsymbol{u}_j\|_{C^{0,\gamma}} + \kappa(\sigma)|z|^{1-\sigma}\,\|\boldsymbol{u}_j\|_{C^{\gamma+\sigma}}\\ &+ \frac{1}{1+\lambda_1+\lambda_0^{-1}}\,\|\boldsymbol{u}_j\|_{C^{1,\gamma}}\,,\end{aligned}$$

for some new constant C. This inequality holds for a single $\boldsymbol{u}_j$. In order to leverage this inequality, we write

$$\begin{aligned}(z+\mathcal{H}M\partial_\theta)\boldsymbol{u}_j &= (z+\mathcal{H}M\partial_\theta)\phi_j\boldsymbol{u}\\ &= \phi_j(z+\mathcal{H}M\partial_\theta)\boldsymbol{u} + \left(\mathcal{H}\left(\phi_j M\partial_\theta\boldsymbol{u}\right) - \phi_j\mathcal{H}\left(M\partial_\theta\boldsymbol{u}\right)\right) + \mathcal{H}\left(M\boldsymbol{u}\partial_\theta\phi_j\right).\end{aligned}$$

Note,

$$\|\phi_j(z+\mathcal{H}M\partial_\theta)\boldsymbol{u}\|_{C^{0,\gamma}} \leq 2\,\|\phi_j\|_{C^{0,\gamma}}\,\|(z+\mathcal{H}M\partial_\theta)\boldsymbol{u}\|_{C^{0,\gamma}}\,.$$

Thus,

$$\begin{aligned}2\left\|\phi_j\right\|_{C^{0,\gamma}}\left\|(z+\mathcal{H}M\partial_\theta)\boldsymbol{u}\right\|_{C^{0,\gamma}} &\geq \left\|\phi_j(z+\mathcal{H}M\partial_\theta)\boldsymbol{u}\right\|_{C^{0,\gamma}} \\ &\geq \left\|(z+\mathcal{H}M\partial_\theta)\phi_j\boldsymbol{u}\right\|_{C^{0,\gamma}} - \left\|[\mathcal{H},\phi_j]\left(M\partial_\theta\boldsymbol{u}\right)\right\|_{C^{0,\gamma}} \\ &- \left\|\mathcal{H}\left(M\boldsymbol{u}\partial_\theta\phi_j\right)\right\|_{C^{0,\gamma}}.\end{aligned}$$

Using the estimate for a single $\boldsymbol{u}_j$ and lemma 3.2.1,

$$\begin{aligned}2\left\|\phi_j\right\|_{C^{0,\gamma}}\left\|(z+\mathcal{H}M\partial_\theta)\boldsymbol{u}\right\|_{C^{0,\gamma}} \geq &\frac{C}{1+\lambda_1}|z|\left\|\phi_j\boldsymbol{u}\right\|_{C^{0,\gamma}} + C\kappa(\sigma)|z|^{1-\sigma}\left\|\phi_j\boldsymbol{u}\right\|_{C^{\gamma+\sigma}} \\ &+ \frac{C}{1+\lambda_1+\lambda_0^{-1}}\left\|\phi_j\boldsymbol{u}\right\|_{C^{1,\gamma}} - C_1\left\|\phi_j\right\|_{C^\infty}\left\|M\right\|_0\left\|\partial_\theta\boldsymbol{u}\right\|_0 \\ &- C_2\left\|\phi_j\right\|_{C^\infty}\left\|M\right\|_{C^{0,\gamma}}\left\|\boldsymbol{u}\right\|_{C^{0,\gamma}},\end{aligned}$$

for some new C_1, C_2. Summing over j, noting that $\|\boldsymbol{u}\| = \left\|\sum_j \phi_j\boldsymbol{u}\right\| \leq \sum_j \|\phi_j\boldsymbol{u}\|$, and choosing σ such that $1 < \sigma+\gamma < 1+\gamma$, we have

$$\begin{aligned}2\left\|(z+\mathcal{H}M\partial_\theta)\boldsymbol{u}\right\|_{C^{0,\gamma}}\sum_j\left\|\phi_j\right\|_{C^{0,\gamma}} \geq &\frac{C}{1+\lambda_0}|z|\left\|\boldsymbol{u}\right\|_{C^{0,\gamma}} + C\kappa(\sigma)|z|^{1-\sigma}\left\|\boldsymbol{u}\right\|_{C^{\gamma+\sigma}} \\ &+ \frac{C}{1+\lambda_1+\lambda_0^{-1}}\left\|\boldsymbol{u}\right\|_{C^{1,\gamma}} \\ &- C_3\left\|M\right\|_{C^{0,\gamma}}\left(\left\|\boldsymbol{u}\right\|_{C^{0,\gamma}} + \left\|\boldsymbol{u}\right\|_{C^{\gamma+\sigma}}\right),\end{aligned}$$

where C_3 depends on $\{\phi_j\}$. If needed, we may push the sector even further out so that

$$\frac{C}{2(1+\lambda_1)}|z| \geq C_3\left\|M\right\|_{C^{0,\gamma}}$$

and

$$C\kappa(\sigma)|z|^{1-\sigma} \geq C_3\left\|M\right\|_{C^{0,\gamma}}.$$

We conclude

$$\begin{aligned}
\|(z+\mathcal{H}M\partial_\theta)\boldsymbol{u}\|_{C^{0,\gamma}} &\geq \frac{C}{\sum_j \|\phi_j\|_{C^{0,\gamma}}}\left(\frac{|z|}{1+\lambda_1}\|\boldsymbol{u}\|_{C^{0,\gamma}}+\frac{1}{1+\lambda_1+\lambda_0^{-1}}\|\boldsymbol{u}\|_{C^{1,\gamma}}\right)\\
&= C\left(\frac{1}{1+\lambda_1}|z|\,\|\boldsymbol{u}\|_{C^{0,\gamma}}+\frac{1}{1+\lambda_1+\lambda_0^{-1}}\|\boldsymbol{u}\|_{C^{1,\gamma}}\right).
\end{aligned}$$

Now, we modify these bounds to accommodate the lower order term $B\boldsymbol{u}$. Using the above, we have

$$\begin{aligned}
\|(z-A)\boldsymbol{u}\|_{C^{0,\gamma}} &\geq \|(z-\Lambda_M)\boldsymbol{u}\|_{C^{0,\gamma}}-\|B\boldsymbol{u}\|_{C^{0,\gamma}} \geq \|(z-\Lambda_M)\boldsymbol{u}\|_{C^{0,\gamma}}-\|B\boldsymbol{u}\|_{C^{\lfloor\beta\rfloor,\beta-\lfloor\beta\rfloor}}\\
&\geq \|(z-\Lambda_M)\boldsymbol{u}\|_{C^{0,\gamma}}-\|B\|\,\|\boldsymbol{u}\|_{C^{0,\gamma}}\\
&\geq \left(\frac{C}{1+\lambda_1}|z|-\|B\|\right)\|\boldsymbol{u}\|_{C^{0,\gamma}}+\frac{C}{1+\lambda_1+\lambda_0^{-1}}\|\boldsymbol{u}\|_{C^{1,\gamma}}
\end{aligned}$$

We may push the sector $S_{\varphi,\omega}$ out even further so that

$$\frac{C}{1+\lambda_1}|z|-\|B\|>0$$

allowing us to conclude that

$$\|(z-A)\boldsymbol{u}\|_{C^{0,\gamma}} \geq C\left(\frac{1}{1+\lambda_1}|z|\,\|\boldsymbol{u}\|_{C^{0,\gamma}}+\frac{1}{1+\lambda_1+\lambda_0^{-1}}\|\boldsymbol{u}\|_{C^{1,\gamma}}\right).$$

□

Finally, we prove the following

Theorem 3.2.3. *For any symmetric positive definite matrix M with component functions $m_{ij} \in C^{0,\gamma}$, there exists a sector $S_{\varphi,\omega}$ with*

$$\left\|(z-A)^{-1}\boldsymbol{u}\right\|_{C^{0,\gamma}} \leq \frac{C(1+\lambda_1)}{|z-\omega|}\|\boldsymbol{u}\|_{C^{0,\gamma}}.$$

where λ_1 is the largest eigenvalue that M achieves over $\theta \in \mathbb{S}^1$.

Proof. In light of the proposition 3.2.2, it suffices to prove that the inverse exists in the sector provided. Let $\boldsymbol{f} \in C^{0,\gamma}$ and let $z \in S_{\varphi,\omega}$. First, consider the problem

$$(z + \mathcal{H}M\partial_\theta)\,\boldsymbol{u} = \boldsymbol{f}.$$

Suppose that for some symmetric positive definite matrix $\tilde{M}$ with $\tilde{m}_{ij} \in C^{0,\gamma}$, that $((z + \mathcal{H}\tilde{M}\partial_\theta)^{-1})$ is well defined. Then, we can rewrite our problem as

$$\left(z + \mathcal{H}\tilde{M}\partial_\theta\right)\boldsymbol{u} + \left(\mathcal{H}\left(M - \tilde{M}\right)\partial_\theta\right)\boldsymbol{u} = \boldsymbol{f}$$

Let $\boldsymbol{v} := (z + \mathcal{H}\tilde{M}\partial_\theta)\boldsymbol{u}$. Thus, our problem is equivalent to solving

$$\boldsymbol{v} + \left(\mathcal{H}(M - \tilde{M})\partial_\theta\right)\left(z + \mathcal{H}\tilde{M}\partial_\theta\right)^{-1}\boldsymbol{v} = \boldsymbol{f}. \tag{3.22}$$

This problem has a unique fixed point whenever

$$\left\|\left(\mathcal{H}(M - \tilde{M})\partial_\theta\right)\left(z + \mathcal{H}\tilde{M}\partial_\theta\right)^{-1}\right\|_{\mathcal{L}(C^{0,\gamma},C^{0,\gamma})} < 1.$$

But,

$$\begin{aligned}&\left\|\left(\mathcal{H}(M - \tilde{M})\partial_\theta\right)\left(z + \mathcal{H}\tilde{M}\partial_\theta\right)^{-1}\right\|_{\mathcal{L}(C^{0,\gamma},C^{0,\gamma})} \\ &\qquad \leq \left\|\mathcal{H}(M - \tilde{M})\partial_\theta\right\|_{\mathcal{L}(C^{1,\gamma},C^{0,\gamma})}\left\|\left(z + \mathcal{H}\tilde{M}\partial_\theta\right)^{-1}\right\|_{\mathcal{L}(C^{0,\gamma},C^{1,\gamma})}\end{aligned}$$

As shown above,

$$\left\|\left(z + \mathcal{H}\tilde{M}\partial_\theta\right)^{-1}\right\|_{\mathcal{L}(C^{0,\gamma},C^{1,\gamma})} \leq C_1$$

for some C_1 which depends on the eigenvalues of $\tilde{M}$ and

$$\left\|\mathcal{H}(M - \tilde{M})\partial_\theta\right\|_{\mathcal{L}(C^{1,\gamma},C^{0,\gamma})} \leq C_2\left\|M - \tilde{M}\right\|_{C^{0,\gamma}}.$$

Thus, (3.22) is solvable provided

$$\left\|M - \tilde{M}\right\|_{C^{0,\gamma}} \leq \frac{C_1}{C_2} \frac{1}{1 + \lambda_1 + \lambda_0^{-1}}.$$

Let $\mathcal{L}_0 = (z + \mathcal{H}\tilde{M}\partial_\theta)$ and $\mathcal{L}_1 = (z + \mathcal{H}M\partial_\theta)$. Since $(z + \mathcal{H}\tilde{M}\partial_\theta)^{-1}$ exists and is well defined for any constant symmetric positive definite $\tilde{M}$, and proposition 3.2.2 gives a shared lower bound on $\|(1-t)\mathcal{L}_0 + t\mathcal{L}_1\|_{\mathcal{L}(C^{1,\gamma}, C^{0,\gamma})}$, we may use the method of continuity to conclude that $(z + \mathcal{H}M\partial_\theta)^{-1}$ exists as well. Further since $(z + \mathcal{H}M\partial_\theta)^{-1}$ exists from $C^{0,\gamma}$ to $C^{1,\gamma}$ we may use the method of continuity again and conclude that $(z - A)^{-1}$ is well defined as well. $\square$

With the above, we have proven that if M is a symmetric positive definite matrix with component functions in $C^{0,\gamma}$, then Λ_M is sectorial and therefore generates an analytic semigroup on $C^{0,\gamma}$. However, in light of proposition 2.3.4, we also have that if M has component functions in $h^{0,\gamma}$ then Λ_M generates an analytic semigroup on $h^{0,\gamma}$ as well.

Proposition 3.2.4. *For any symmetric positive definite matrix M with component functions in $h^{0,\gamma}$, there exists a sector $S_{\varphi,\omega}$ such that*

$$\left\|(z - A)^{-1}\boldsymbol{u}\right\|_{h^{0,\gamma}} \leq \frac{C}{|z - \omega|} \|\boldsymbol{u}\|_{h^{0,\gamma}}$$

Proof. Given the fact that the norm on the little Hölder continuous spaces is the same as that for the big Hölder continuous spaces, the previous proposition can be repeated in the little Hölder continuous spaces provided that the operator $(z - A)$ is invertible from $h^{0,\gamma}$ to $h^{1,\gamma}$ for $z \in S_{\varphi,\omega}$. So, let $\boldsymbol{f} \in h^{0,\gamma}$ and consider the problem

$$(z - A)\boldsymbol{u} = \boldsymbol{f}.$$

Since $h^{0,\gamma} \subset C^{0,\gamma}$, we know that there exists some $\boldsymbol{u} \in C^{1,\gamma}$ such that $(z - A)\boldsymbol{u} = \boldsymbol{f}$. We need only know that $\boldsymbol{u} \in h^{1,\gamma}$. Since $\boldsymbol{u} \in C^{1,\gamma}$, we have that $\boldsymbol{u} \in h^{0,\gamma}$ as well. Let $\boldsymbol{g} := z\boldsymbol{u} - B\boldsymbol{u} - \boldsymbol{f} \in h^{0,\gamma}$. Thus,

$$\mathcal{H}M\partial_\theta \boldsymbol{u} = \boldsymbol{g} \in h^{0,\gamma},$$

implies that

$$\partial_\theta \boldsymbol{u} = -M^{-1}\mathcal{H}\boldsymbol{g} \in h^{0,\gamma}$$

since M is symmetric positive definite, M^{-1} is well defined with component functions in $h^{0,\gamma}$ so that $\boldsymbol{u} \in h^{1,\gamma}$ as desired. $\square$

We summarize our generation results in the theorem below.

Theorem 3.2.5. *Given a symmetric positive definite matrix M with component functions $m_{ij} \in C^{0,\gamma}$, we have the following:*

(i.) The operator

$$\Lambda_M = -\mathcal{H}\left(M\partial_\theta\right) : C^{1,\gamma} \mapsto C^{0,\gamma}$$

generates an analytic semigroup on $C^{0,\gamma}$. Further, if $B : C^\gamma \mapsto C^\beta$ for any $0 < \gamma < \beta$, then

$$A := \Lambda_M + B \tag{3.23}$$

generates an analytic semigroup on $C^{0,\gamma}$.

(ii.) If in addition $m_{ij} \in h^{0,\gamma}$, then Λ_M generates an analytic semigroup on $h^{0,\gamma}$. Also, if $B : h^\gamma \mapsto C^\beta$ for any $0 < \gamma < \beta$, then $A := \Lambda_M + B$ generates an analytic semigroup on $h^{0,\gamma}$.

To prove local wellposedness in the nonlinear setting, we will need to make sense of some abstract spaces defined by the operator A. Suppose that we have some Banach space X such that $A : D(A) \subset X \mapsto X$. Let us define the spaces $D_A(\sigma, \infty)$ and $D_A(\sigma)$ as the interpolation spaces

$$D_A(\sigma, \infty) = (X, D(A))_{\sigma,\infty} \tag{3.24}$$

and

$$D_A(\sigma) = (X, D(A))_\sigma. \tag{3.25}$$

Using these spaces, let us also define

$$\begin{aligned} D_A(\sigma+1,\infty) &= \{x \in D(A) : Ax \in D_A(\sigma,\infty)\} \\ D_A(\sigma+1) &= \{x \in D_A(\sigma+1,\infty) : Ax \in D_A(\sigma)\}. \end{aligned} \tag{3.26}$$

We are free to choose both $D(A)$ and X as we wish, provided of course that A is defined on $D(A)$. Thus, we can make the choice that $D(A) = C^{1,\alpha}$ and $X = C^{0,\alpha}$. In this case, it is obvious that for any $\alpha < \gamma < 1$ there is some σ so that

$$D_A(\sigma,\infty) = (C^{0,\alpha}, C^{1,\alpha})_{\sigma,\infty} \simeq C^{0,\gamma}.$$

While it is clear that $C^{1,\gamma} \subset D_A(\sigma+1,\infty)$, it is not immediately clear that the two spaces are actually equal.

Proposition 3.2.6. *Let $A = \Lambda_M + B$ with $A : D(A) = C^{1,\alpha} \mapsto C^{0,\alpha}$ be as in Theorem 3.2.5, for matrix M which has component functions in $C^{0,\gamma}$ with $\alpha < \gamma < 1$. Then, there is some $\sigma \in (0,1)$ such that $D_A(\sigma,\infty) \simeq C^{0,\gamma}$ and $D_A(\sigma+1,\infty) \simeq C^{1,\gamma}$.*

Proof. It is immediately clear that there exists some $\sigma \in (0,1)$ so that $D_A(\sigma,\infty) = C^{0,\gamma}$. We will show that for this same σ, $D_A(\sigma+1,\infty) = C^{1,\gamma}$. It is obvious that if $\boldsymbol{u} \in C^{1,\gamma}$ then $\boldsymbol{u} \in D_A(\sigma+1,\infty)$ as $A : C^{1+\gamma} \mapsto C^{0,\gamma}$ by assumption. All that is left to show is that if $\boldsymbol{u} \in D_A(\sigma+1,\infty)$ then $\boldsymbol{u} \in C^{1,\gamma}$. So, let $\boldsymbol{u} \in D_A(\sigma+1,\infty)$. Then, $\boldsymbol{u} \in C^{1,\alpha}$ and $A\boldsymbol{u} \in C^{0,\gamma}$. Since $\boldsymbol{u} \in C^{1,\alpha}$, we have $\boldsymbol{u} \in C^{0,\gamma}$ as well. By proposition 3.2.5, there is some sector $S_{\varphi,\omega}$ such that for all $z \in S_{\varphi,\omega}$,

$$\|\boldsymbol{u}\|_{C^{1,\gamma}} \leq C \, \|(z-A)\boldsymbol{u}\|_{C^{0,\gamma}} \, ,$$

where constant C depends on M and the sector. Thus,

$$\|\boldsymbol{u}\|_{C^{1,\gamma}} \leq C \, \|(z-A)\boldsymbol{u}\|_{C^{0,\gamma}} \leq C \left(\|A\boldsymbol{u}\|_{C^{0,\gamma}} + \|\boldsymbol{u}\|_{C^{0,\gamma}} \right),$$

for some new constant C which depends on z. $\square$

We prove a similar result for the spaces $D_A(\sigma)$ and $D_A(\sigma+1)$.

Proposition 3.2.7. *Let $A = \Lambda_M + B$ with $A : D(A) = h^{1,\alpha} \mapsto h^{0,\alpha}$ be as in Theorem 3.2.5, with M having component functions in $h^{0,\gamma}$ with $\alpha < \gamma < 1$. Then there is some $\sigma \in (0,1)$ such that $D_A(\sigma) \simeq h^{0,\gamma}$ and $D_A(\sigma+1) \simeq h^{1,\gamma}$.*

Proof. It is easy to show that $h^{1,\gamma} \subset D_A(\sigma+1)$. In light of the previous proposition, we have that $D_A(\sigma+1) \subset C^{1,\gamma}$. Further, we have already shown in proposition 3.2.4 that there is some complex number z for which $(z-A)$ is an invertible operator from $h^{0,\gamma}$ to $h^{1,\gamma}$ with appropriate bounds. Thus, for this z,

$$\|\boldsymbol{u}\|_{h^{1,\gamma}} \leq \|(z-A)\boldsymbol{u}\|_{h^{0,\gamma}} \leq C\left(\|A\boldsymbol{u}\|_{h^{0,\gamma}} + \|\boldsymbol{u}\|_{h^{0,\gamma}}\right),$$

for C depending on z. Thus $D_A(\sigma+1) \simeq h^{1,\gamma}$.

$\square$

Chapter 4

Semilinear Peskin Problem

We first address the case for a simple elasticity law $\boldsymbol{F}(\boldsymbol{X}) = \partial_\theta^2 \boldsymbol{X}$. After integrating by parts, the evolution of the boundary is determined by

$$\begin{aligned}\partial_t \boldsymbol{X} = &\frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'}\left(\log|\Delta \boldsymbol{X}|\right)\partial_{\theta'}\boldsymbol{X}' d\theta' \\ &- \frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'}\left(\frac{\Delta \boldsymbol{X}\otimes \Delta \boldsymbol{X}}{|\Delta \boldsymbol{X}|^2}\right)\partial_{\theta'}\boldsymbol{X}' d\theta'.\end{aligned}$$

We can rewrite this as

$$\partial_t \boldsymbol{X} = -\frac{1}{4\pi}\mathcal{H}\left(\partial_\theta \boldsymbol{X}\right) + \mathcal{R}(\boldsymbol{X})$$

where

$$\begin{aligned}\mathcal{R}(\boldsymbol{X}) = &\frac{1}{4\pi}\int_{\mathbb{S}^1}\left(\partial_{\theta'}\log|\Delta \boldsymbol{X}| + \frac{1}{2}\cot\left(\frac{\theta-\theta'}{2}\right)\right)\partial_{\theta'}\boldsymbol{X}' d\theta' \\ &- \frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'}\left(\frac{\Delta \boldsymbol{X}\otimes \Delta \boldsymbol{X}}{|\Delta \boldsymbol{X}|^2}\right)\partial_{\theta'}\boldsymbol{X}' d\theta'.\end{aligned}$$

We will first prove that this problem is locally well-posed. Do do this, we will construct a contraction on a subset of $C([0,T]; C^{1,\gamma})$ for some $T > 0$. From there, we will prove higher spatial regularity of solutions and use this to show that the BI, IB and jump formulations of the semilinear Peskin problem coincide. In Section 4.3, using the jump formulaton, we will compute the equilibria of the system and then use the BI formulation to prove stability about them. We then computationally verify these results in Section 4.3.4. Finally, we will state some characteristics of global in time solutions.

4.1 Local Well-Posedness

Consider the term $\mathcal{R}$ in as above. We may write

$$\mathcal{R} = \mathcal{R}_C + \mathcal{R}_T, \tag{4.1}$$

where

$$\mathcal{R}_C(\boldsymbol{X})(\theta) = -\frac{1}{4\pi}\int_{\mathbb{S}^1} K_C(\theta,\theta')\partial_{\theta'}\boldsymbol{X}'d\theta', \tag{4.2}$$

$$K_C(\theta,\theta') = \frac{\Delta\boldsymbol{X}\cdot\partial_{\theta'}\boldsymbol{X}'}{|\Delta\boldsymbol{X}|^2} - \frac{1}{2}\cot\left(\frac{\theta-\theta'}{2}\right), \tag{4.3}$$

and

$$\mathcal{R}_T(\boldsymbol{X})(\theta) = -\frac{1}{4\pi}\int_{\mathbb{S}^1} K_T(\theta,\theta')\partial_{\theta'}\boldsymbol{X}'d\theta', \tag{4.4}$$

$$K_T(\theta,\theta') = \partial_\theta'\left(\frac{\Delta\boldsymbol{X}\otimes\Delta\boldsymbol{X}}{|\Delta\boldsymbol{X}|^2}\right). \tag{4.5}$$

Before we can construct a contraction, we need to first show that $\mathcal{R}$ is well behaved. For this purpose, let us define operators

$$F_C\boldsymbol{u} = \int_{\mathbb{S}^1} K_C(\theta,\theta')\boldsymbol{u}(\theta')d\theta'$$

and

$$F_T\boldsymbol{u} = \int_{\mathbb{S}^1} K_T(\theta,\theta')\boldsymbol{u}(\theta')d\theta'.$$

A straightforward application of proposition 2.3.1 shows that for $\boldsymbol{X}\in C^{1,\gamma}$, $F_T\boldsymbol{u}\in C^{\lfloor 2\gamma\rfloor,2\gamma-\lfloor 2\gamma\rfloor}$. Further, proposition 2.3.2 directly shows that $F_C\boldsymbol{u}\in C^{\lfloor 2\gamma\rfloor,2\gamma-\lfloor 2\gamma\rfloor}$. Thus, we conclude that

Proposition 4.1.1. *If* $\boldsymbol{X}\in C^{1,\gamma}(\mathbb{S}^1)$ *with* $\gamma\in(0,1)$ *and* $|\boldsymbol{X}|_* > 0$ *and* *(i) if $\gamma \neq 1/2$ then $F_C\boldsymbol{u} \in C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}(\mathbb{S}^1)$ with*

$$\|F_C\boldsymbol{u}\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} \leq C\frac{\|\boldsymbol{X}\|_{C^{1,\gamma}}^3}{|\boldsymbol{X}|_*^3}\|\boldsymbol{u}\|_{C^{0,\gamma}}.$$

(ii) if $\gamma = 1/2$ then $\mathcal{R}_C(\boldsymbol{X}) \in C^{0,\alpha}(\mathbb{S}^1)$ for any $\alpha \in (0,1)$ with

$$\|F_C\boldsymbol{u}\|_{C^{0,\alpha}} \leq C\frac{\|\boldsymbol{X}\|_{C^{1,\gamma}}^3}{|\boldsymbol{X}|_*^3}\|\boldsymbol{u}\|_{C^{0,\gamma}}.$$

In the above, the constant C does not depend on $\boldsymbol{X}$ or $\boldsymbol{u}$.

Similarly, for F_T we have

Proposition 4.1.2. *Suppose $\boldsymbol{X} \in C^{1,\gamma}(\mathbb{S}^1)$ with $\gamma \in (0,1)$ and $|\boldsymbol{X}|_* > 0$. We have the following estimates.*

(i) if $\gamma \neq 1/2$ then $F_T\boldsymbol{u} \in C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}$ with

$$\|\mathcal{R}_T(\boldsymbol{X})\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} \leq C\frac{\|\boldsymbol{X}\|_{C^{1,\gamma}}^3}{|\boldsymbol{X}|_*^3}\|\boldsymbol{u}\|_{C^{0,\gamma}}.$$

(ii) if $\gamma = 1/2$ then $F_T\boldsymbol{u} \in C^{0,\alpha}$ for any $\alpha \in (0,1)$ with

$$\|F_T\boldsymbol{u}\|_{C^{0,\alpha}} \leq C\frac{\|\boldsymbol{X}\|_{C^{1,\gamma}}^3}{|\boldsymbol{X}|_*^3}\|\boldsymbol{u}\|_{C^{0,\gamma}}.$$

In fact, it isn't hard to see that we have a slightly stronger bound. We can actually write

$$\|F_T\boldsymbol{u}\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} \leq C\frac{\|\partial_\theta \boldsymbol{X}\|_{C^{0,\gamma}}^3}{|\boldsymbol{X}|_*^3}\|\boldsymbol{u}\|_{C^{0,\gamma}} \tag{4.6}$$

The fact that we have for $\boldsymbol{X} \in C^{1,\gamma}$ $\mathcal{R}(\boldsymbol{X}) \in C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}$ while $\Lambda \boldsymbol{X} \in C^{0,\gamma}$ cements our characterization of this problem as semilinear. With this in mind, we now set to work on our contraction.

We are looking for a mild solution of the Peskin system which is a solution of

$$\boldsymbol{X}(t) = e^{t\Lambda}\boldsymbol{X}_0 + \int_0^t e^{(t-s)\Lambda}\mathcal{R}(\boldsymbol{X}(s))ds,$$

where $\boldsymbol{X} : \mathbb{S}^1 \to \mathbb{R}^2$ is a closed curve. To address the local existence of a solution, we will show that the map

$$S(\boldsymbol{X}, t; \boldsymbol{X}_0) := e^{\Lambda t}\boldsymbol{X}_0 + \int_0^t e^{(t-s)\Lambda}\mathcal{R}(\boldsymbol{X}(s))ds \tag{4.7}$$

has a fixed point in a certain subset of $C([0,T]; C^{1,\gamma}(\mathbb{S}^1))$ for suitably chosen values of T. We will use

Theorem 4.1.3 (Banach Fixed Point Theorem)**.** *Let (U, d) be a non-empty complete metric space. If $S : U \to U$ is an operator with $d(Su, Sv) \leq qd(u, v)$ for $q \in [0, 1)$, then S has a unique fixed point $u^* \in U$.*

Given how our previous estimates on $\mathcal{R}(\boldsymbol{X})$ rely on $|\boldsymbol{X}|_* > 0$, we take a subset of $C^{1,\gamma}(\mathbb{S}^1)$ which includes only $\boldsymbol{Y} \in C^{1,\gamma}$ with $|\boldsymbol{Y}|_* \geq m > 0$. We define our set as follows.

Proposition 4.1.4 (Adapted from [22] proposition 8.7)**.** *For any $M > m > 0$, the set $O^{M,m} := \{\boldsymbol{X} \in C^{1,\gamma}(\mathcal{D}) : \|\boldsymbol{X}\|_{C^{1,\gamma}} \leq M$ and $|\boldsymbol{X}|_* \geq m\}$ is closed in $C^{1,\gamma}(\mathbb{S}^1)$. By extension, for any $T \geq t \geq 0$ the set $O_t^{M,m} = \{\boldsymbol{X} \in C([0,t]; C^{1,\gamma}(\mathbb{S}^1)) : \boldsymbol{X}(s) \in O^{M,m}$ for all $s \in [0,t]\}$ is closed in $C([0,T]; C^{1,\gamma}(\mathbb{S}^1))$.*

Proof. For $\boldsymbol{X}, \boldsymbol{Y} \in C^{1,\gamma}(\mathbb{S}^1)$, we have by the reverse triangle inequality

$$\begin{aligned} \left| |\boldsymbol{X}|_* - |\boldsymbol{Y}|_* \right| &= \left| \inf_{\theta\neq\theta'} \frac{|\boldsymbol{X} - \boldsymbol{X}'|}{|\theta - \theta'|} - \inf_{\theta\neq\theta'} \frac{|\boldsymbol{Y} - \boldsymbol{Y}'|}{|\theta - \theta'|} \right| \leq \sup_{\theta\neq\theta'} \left| \frac{|\boldsymbol{X} - \boldsymbol{X}'|}{|\theta - \theta'|} - \frac{|\boldsymbol{Y} - \boldsymbol{Y}'|}{|\theta - \theta'|} \right| \\ &\leq \sup_{\theta\neq\theta'} \frac{|\boldsymbol{X} - \boldsymbol{Y} - (\boldsymbol{X}' - \boldsymbol{Y}')|}{|\theta - \theta'|} \leq \|\boldsymbol{X} - \boldsymbol{Y}\|_{C^{1,\gamma}} \end{aligned}$$

Also by the reverse triangle inequality,

$$\left| \|\boldsymbol{X}\|_{C^{1,\gamma}} - \|\boldsymbol{Y}\|_{C^{1,\gamma}} \right| \leq \|\boldsymbol{X} - \boldsymbol{Y}\|_{C^{1,\gamma}} .$$

Therefore the maps $|\cdot|_* : C^{1,\gamma} \to [0, \infty)$ and $\|\cdot\|_{C^{1,\gamma}} : C^{1,\gamma} \to [0, \infty)$ are continuous. Since $O^{M,m}$ is the intersection of preimages of two closed sets under continuous maps, it is closed in $C^{1,\gamma}$. Since $[0, t]$ is closed in $[0, T]$, the second statement follows. $\square$

To show that the map $S(\boldsymbol{X}, t; \boldsymbol{X}_0)$ is a contraction over the above set, we first prove the following:

Proposition 4.1.5. *For any $M > m > 0$ and $\gamma \in (0,1)$, the remainder term $\mathcal{R}(\boldsymbol{X})$ is Lipschitz on any convex set $\mathcal{B} \subset O^{M,m}$ and*

(i) if $\gamma \neq 1/2$, $\mathcal{R} : \mathcal{B} \to C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}(\mathbb{S}^1)$ with

$$\|\mathcal{R}(\boldsymbol{X}) - \mathcal{R}(\boldsymbol{Y})\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} \leq C \frac{M^4}{m^4} \|\boldsymbol{X} - \boldsymbol{Y}\|_{C^{1,\gamma}} .$$

(ii) if $\gamma = 1/2$, then for any $\alpha \in (0,1)$, $\mathcal{R} : \mathcal{B} \to C^{0,\alpha}(\mathbb{S}^1)$ with

$$\|\mathcal{R}(\boldsymbol{X}) - \mathcal{R}(\boldsymbol{Y})\|_{C^{0,\alpha}} \leq C \frac{M^4}{m^4} \|\boldsymbol{X} - \boldsymbol{Y}\|_{C^{1,\gamma}} .$$

Proof. We show only the $\gamma \neq 1/2$ case as the $\gamma = 1/2$ case follows from the arguments when $\gamma < 1/2$. Let $M > m > 0$ and $\mathcal{B} \subset O^{M,m}$ be convex. It suffices to show that the linearization of $\mathcal{R}$, $\partial_{\boldsymbol{X}} \mathcal{R}(\boldsymbol{X})$, is bounded on $C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}(\mathbb{S}^1)$ for any $\boldsymbol{X} \in \mathcal{B}$ since

$$\begin{aligned} \|\mathcal{R}(\boldsymbol{X}) - \mathcal{R}(\boldsymbol{Y})\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} &= \left\| \int_0^1 \frac{d}{ds} \mathcal{R}((1-s)\boldsymbol{X} + s\boldsymbol{Y}) \right\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} ds \\ &\leq \int_0^1 \|\partial_{\boldsymbol{X}} \mathcal{R}((1-s)\boldsymbol{X} + s\boldsymbol{Y})\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} ds \, \|\boldsymbol{X} - \boldsymbol{Y}\|_{C^{1,\gamma}} . \end{aligned}$$

We can prove that $\partial_{\boldsymbol{X}} \mathcal{R} = \partial_{\boldsymbol{X}} \mathcal{R}_C + \partial_{\boldsymbol{X}} \mathcal{R}_T$ is bounded on $C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}$ by showing that both $\partial_{\boldsymbol{X}} \mathcal{R}_C$ and $\partial_{\boldsymbol{X}} \mathcal{R}_T$ are. When we linearize $\mathcal{R}_C$ we get for any constant vectors $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$

$$\begin{aligned} \partial_{\boldsymbol{X}} \mathcal{R}_C(\boldsymbol{X}) \boldsymbol{Z} &= \frac{1}{4\pi} \int_{\mathbb{S}^1} \left(\frac{\Delta \boldsymbol{X} \cdot \partial_{\theta'} \boldsymbol{X}'}{|\Delta \boldsymbol{X}|^2} - \frac{1}{2} \cot\left(\frac{\theta - \theta'}{2} \right) \right) \left(\boldsymbol{V}_1 - \partial_{\theta'} \boldsymbol{Z}' \right) d\theta' \\ &+ \frac{1}{4\pi} \int_{\mathbb{S}^1} \partial_{\theta'} \left(\partial_{\boldsymbol{X}} \left(\log |\Delta \boldsymbol{X}| \right) \boldsymbol{Z} \right) \left(\boldsymbol{V}_2 - \partial_{\theta'} \boldsymbol{X}' \right) d\theta', =: R_1 + R_2 \end{aligned}$$

where $\partial_{\boldsymbol{X}} \left(\log |\Delta \boldsymbol{X}| \right) \boldsymbol{Z}$ is the Frechét derivative of $\log |\Delta \boldsymbol{X}|$ with pertubation $\boldsymbol{Z}$. From proposition 2.3.2 we know that

$$\|R_1\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} \leq C \frac{\|\boldsymbol{X}\|_{C^{1,\gamma}}^3}{|\boldsymbol{X}|_*^3} \|\boldsymbol{Z}\|_{C^{1,\gamma}} .$$

So, we turn to R_2. However, lemma 2.2.8 implies that $\partial_{\boldsymbol{X}} \left(\log |\Delta \boldsymbol{X}| \right) \boldsymbol{Z} \in \mathcal{S}^H_{0,1,\gamma}$. Indeed,

$$\partial_{\boldsymbol{X}}\left(\log|\Delta\boldsymbol{X}|\right)\boldsymbol{Z} = \frac{\Delta\boldsymbol{X}\cdot\Delta\boldsymbol{Z}}{|\Delta\boldsymbol{X}|^2}.$$

Thus, since this operator acts component wise, we may apply proposition 2.3.1 and find

$$\left|\int_{\mathbb{S}^1}\partial_{\theta'}\left(\partial_{\boldsymbol{X}}\left(\log|\Delta\boldsymbol{X}|\right)\boldsymbol{Z}\right)\partial_{\theta'}\boldsymbol{X}'d\theta'\right| \le C\frac{\|\boldsymbol{X}\|_{C^{1,\gamma}}^4}{|\boldsymbol{X}|_*^4}\|\boldsymbol{Z}\|_{C^{1,\gamma}}.$$

Since $\boldsymbol{X}\in O^{M,m}$ we conclude that

$$\|\partial_{\boldsymbol{X}}\mathcal{R}(\boldsymbol{X})\boldsymbol{Z}\|_{C^{\lfloor 2\gamma\rfloor,2\gamma-\lfloor 2\gamma\rfloor}} \le C\frac{M^4}{m^4}\|\boldsymbol{Z}\|_{C^{1,\gamma}}.$$

We can do the same for $\mathcal{R}_T$. Indeed,

$$\begin{aligned}\partial_{\boldsymbol{X}}\mathcal{R}_T(\boldsymbol{X})\boldsymbol{Z} = &-\frac{1}{4\pi}\int_{\mathbb{S}^1}\partial_{\theta'}\left(\frac{\Delta\boldsymbol{X}\otimes\Delta\boldsymbol{X}}{|\Delta\boldsymbol{X}|}^2\right)\partial_{\theta'}\boldsymbol{Z}'d\theta' \\ &-\frac{1}{4\pi}\int_{\mathbb{S}^1}\partial_{\theta'}\left(\partial_{\boldsymbol{X}}\left(\frac{\Delta\boldsymbol{X}\otimes\Delta\boldsymbol{X}}{|\Delta\boldsymbol{X}|}\right)\boldsymbol{Z}\right)\partial_{\theta'}\boldsymbol{X}d\theta' =: R_1+R_2.\end{aligned}$$

A straightforward application of proposition 2.3.1 shows that

$$\|R_1\|_{C^{\lfloor 2\gamma\rfloor,2\gamma-\lfloor 2\gamma\rfloor}} \le C\frac{\|\boldsymbol{X}\|_{C^{1,\gamma}}^3}{|\boldsymbol{X}|_*^3}\|\boldsymbol{Z}\|_{C^{1,\gamma}}.$$

For R_2, note that lemma 2.2.6 guarantees that $\partial_{\boldsymbol{X}}\left(\frac{\Delta\boldsymbol{X}\otimes\Delta\boldsymbol{X}}{|\Delta\boldsymbol{X}|}\right)\boldsymbol{Z}$ is a sum of terms, each of which is in $\mathcal{S}^H_{0,1,\gamma}$. Thus, applying proposition 2.3.1 to each of these terms implies that $R_2\in C^{\lfloor 2\gamma\rfloor,2\gamma-\lfloor 2\gamma\rfloor}$. Further, tracking exponents given in lemma 2.2.6, and noting that $\|\boldsymbol{X}\|_{C^{1,\gamma}} > |\boldsymbol{X}|_*$, we conclude

$$\|R_2\|_{C^{\lfloor 2\gamma\rfloor,2\gamma-\lfloor 2\gamma\rfloor}} \le C\frac{\|\boldsymbol{X}\|_{C^{1,\gamma}}^4}{|\boldsymbol{X}|_*^4}\|\boldsymbol{Z}\|_{C^{1,\gamma}}.$$

Since $\boldsymbol{X}\in O^{M,m}$ we have

$$\mathcal{R}_T \leq C\frac{M^4}{m^4}\left\|\boldsymbol{Z}\right\|_{C^{1,\gamma}}$$

as desired.

□

Consider a set of functions with

$$\left\|\boldsymbol{X}-\boldsymbol{X}_0\right\|_{C^{1,\gamma}} \leq \frac{1}{2}\left|\boldsymbol{X}_0\right|_* .$$

In this set, we have

$$\begin{aligned}\left|\boldsymbol{X}\right|_* \geq \left|\boldsymbol{X}_0\right|_* - \left\|\boldsymbol{X}-\boldsymbol{X}_0\right\|_{C^{1,\gamma}} \geq \frac{1}{2}\left|\boldsymbol{X}_0\right|_* = m,\\ \left\|\boldsymbol{X}\right\|_{C^{1,\gamma}} \leq \left\|\boldsymbol{X}_0\right\|_{C^{1,\gamma}} + \frac{1}{2}\left|\boldsymbol{X}_0\right|_* = M.\end{aligned}$$

With this in mind, define the set

$$\mathcal{B}_T = \left\{\boldsymbol{X}\in C([0,T];C^{1,\gamma}(\mathbb{S}^1)) : \left\|\boldsymbol{X}-\boldsymbol{X}_0\right\|_{C([0,T];C^{1,\gamma})} \leq \frac{1}{2}\left|\boldsymbol{X}_0\right|_*\right\}, \tag{4.8}$$

where we have abused notation slightly to write $\boldsymbol{X}_0$ as the function that is constant in time taking value $\boldsymbol{X}_0$. Note that $\mathcal{B}_T$ is a convex set in $O_T^{M,m}$. We are now in position to prove the existence of a mild solution to the Peskin system.

Proposition 4.1.6. *There exists some time $T>0$ such that $S(\boldsymbol{X},t;\boldsymbol{X}_0)$ forms a contraction on $\mathcal{B}_T$.*

Proof. We first show that there is some $T_1>0$ for which S maps B_{T_1} to itself. For $\boldsymbol{X}\in\mathcal{B}_{T_1}$ and $\gamma\neq 1/2$,

$$\begin{aligned}\left\|S(\boldsymbol{X},t;\boldsymbol{X}_0)-\boldsymbol{X}_0\right\|_{C^{1,\gamma}} &\leq \left\|e^{t\Lambda}\boldsymbol{X}_0-\boldsymbol{X}_0\right\|_{C^{1,\gamma}} + \int_0^t \left\|e^{(t-s)\Lambda}\mathcal{R}(\boldsymbol{X}(s))\right\|_{C^{1,\gamma}} ds\\ &\leq \left\|(e^{t\Lambda}-1)\boldsymbol{X}_0\right\|_{C^{1,\gamma}} + C\int_0^t (t-s)^{\gamma-1}\left\|\mathcal{R}(\boldsymbol{X}(s))\right\|_{C^{\lfloor 2\gamma\rfloor,2\gamma-\lfloor 2\gamma\rfloor}} ds\\ &\leq \left\|(e^{t\Lambda}-1)\boldsymbol{X}_0\right\|_{C^{1,\gamma}} + C\frac{M^5}{m^4}\int_0^t (t-s)^{\gamma-1}ds\\ &\leq \left\|(e^{t\Lambda}-1)\boldsymbol{X}_0\right\|_{C^{1,\gamma}} + Ct^{\gamma}\frac{M^5}{m^4},\end{aligned} \tag{4.9}$$

where we used equation (3.14) between the first and second line and Propostions 4.1.1 and 4.1.2 between lines two and three. Since $\boldsymbol{X}_0 \in h^{1,\gamma}(\mathbb{S}^1)$, by Proposition 3.1.11, we may take T_1 small enough so that

$$\left\|(e^{t\Lambda} - 1)\boldsymbol{X}_0\right\|_{C^{1,\gamma}} + Ct^{\gamma}\frac{M^5}{m^4} \leq \frac{1}{2}\left|\boldsymbol{X}_0\right|_*$$

for all $0 \leq t \leq T_1$. We now show that S forms a contraction on $\mathcal{B}_{T_2}$ for some $T_2 > 0$. Let $\boldsymbol{X}$, $\boldsymbol{Y} \in \mathcal{B}_t$. Then,

$$\begin{aligned}
\|S(\boldsymbol{X}, t; \boldsymbol{X}_0) - S(\boldsymbol{Y}, t; \boldsymbol{X}_0)\|_{C^{1,\gamma}} &\leq \int_0^t \left\|e^{(t-s)\Lambda}[\mathcal{R}(\boldsymbol{X}(s)) - \mathcal{R}(\boldsymbol{Y}(s))]\right\|_{C^{1,\gamma}} \\
&\leq C\int_0^t (t-s)^{\gamma-1}\left\|\mathcal{R}(\boldsymbol{X}(s)) - \mathcal{R}(\boldsymbol{Y}(s))\right\|_{C^{\lfloor 2\gamma\rfloor, 2\gamma - \lfloor 2\gamma\rfloor}} ds \\
&\leq C\frac{M^4}{m^4}\left\|\boldsymbol{X} - \boldsymbol{Y}\right\|_{C([0,t];C^{1,\gamma})}\int_0^t (t-s)^{\gamma-1} ds \\
&\leq Ct^{\gamma}\frac{M^4}{m^4}\left\|\boldsymbol{X} - \boldsymbol{Y}\right\|_{C([0,t];C^{1,\gamma})},
\end{aligned}$$

where Proposition 4.1.5 was used between lines two and three. There exists some time $T_2 > 0$ such that

$$Ct^{\gamma}\frac{M^4}{m^4} \leq \frac{1}{2}$$

for all $0 \leq t \leq T_2$. Taking $T = \min\{T_1, T_2\}$ gives the desired result. In the case of $\gamma = 1/2$, we use the second statements in Propositions 4.1.1, 4.1.2 and 4.1.5 with the choice of $\alpha = 3\gamma/2$ and the result follows from the same arguments.

$\square$

Proposition 4.1.6 gives the local existence of a solution to the Peskin problem for any initial data $\boldsymbol{X}_0 \in h^{1,\gamma}(\mathbb{S}^1)$ with $\left|\boldsymbol{X}_0\right|_* > 0$. Our next result shows that our mild solution is, in fact, a solution to the differential form of the PDE away from the initial time. In particular, we have

Lemma 4.1.7. *Let $\boldsymbol{X}(t) \in C([0,T]; C^{1,\gamma}(\mathbb{S}^1))$ be a mild solution with initial data $\boldsymbol{X}_0$ as in the assumptions in Theorem 1.2.3, then*

$$\partial_t \boldsymbol{X} = \Lambda \boldsymbol{X} + \mathcal{R}(\boldsymbol{X}) \tag{4.10}$$

for the time interval $(0,t)$. Furthermore, $\partial_t \boldsymbol{X} \in C([0,T]; C^{0,\gamma}(\mathbb{S}^1))$.

Proof. We use some of the ideas of the proof of Lemma 4.1.6 in [21]. First, we claim that $\int_0^t \boldsymbol{X}(s)ds \in C^{1,\gamma}$ and

$$\boldsymbol{X}(t) = \boldsymbol{X}_0 + \Lambda \int_0^t \boldsymbol{X}(s)ds + \int_0^t \mathcal{R}(\boldsymbol{X}(s))ds \tag{4.11}$$

for all $[0,t]$. The first fact follows from $\left\|\int_0^t \boldsymbol{X}(s)ds\right\|_{C^{1,\gamma}} \leq t \sup_{0\leq s\leq t} \|\boldsymbol{X}(s)\|_{C^{1,\gamma}} \leq Ct$.

To show (4.11) we note from (3.10) that $\left\|e^{(s-\sigma)\Lambda}\mathcal{R}(\boldsymbol{X}(\sigma))\right\|_{C^{1,\gamma}} \leq \|\mathcal{R}(\boldsymbol{X}(\sigma)\|_{C^{1,\gamma}} \leq C(\|\boldsymbol{X}(\sigma)\|_{C^{1,\gamma}}, |\boldsymbol{X}(\sigma)|_*)$ for $0 \leq \sigma \leq s \leq t$, and so by Fubini,

$$\begin{aligned}\int_0^t \boldsymbol{X}(s)ds &= \int_0^t e^{s\Lambda}\boldsymbol{X}_0 ds + \int_0^t \int_0^s e^{(s-\sigma)\Lambda}\mathcal{R}(\boldsymbol{X}(\sigma))d\sigma ds \\ &= \int_0^t e^{s\Lambda}\boldsymbol{X}_0 ds + \int_0^t \int_\sigma^t e^{(s-\sigma)\Lambda}\mathcal{R}(\boldsymbol{X}(\sigma))ds d\sigma.\end{aligned} \tag{4.12}$$

From the construction of the Λ operator in subsection 3.1.1, we can write $\Lambda e^{t\Lambda} = \partial_t(e^{t\Lambda})$. Therefore

$$\begin{aligned}\int_0^t \Lambda\boldsymbol{X}(s)ds &= \int_0^t \Lambda e^{s\Lambda}\boldsymbol{X}_0 ds + \int_0^t \int_0^s \Lambda e^{(s-\sigma)\Lambda}\mathcal{R}(\boldsymbol{X}(\sigma))d\sigma ds \\ &= \int_0^t \Lambda e^{s\Lambda}\boldsymbol{X}_0 ds + \int_0^t \int_\sigma^t \Lambda e^{(s-\sigma)\Lambda}\mathcal{R}(\boldsymbol{X}(\sigma))ds d\sigma \\ &= (e^{t\Lambda}-1)\boldsymbol{X}_0 + \int_0^t \left(e^{(t-\sigma)\Lambda}-1\right)\mathcal{R}(\boldsymbol{X}(\sigma))d\sigma \\ &= \boldsymbol{X}(t) - \boldsymbol{X}_0 - \int_0^t \mathcal{R}(\boldsymbol{X}(\sigma))d\sigma\end{aligned}$$

and hence (4.11).

2. Since $\int_0^t \boldsymbol{X}(s)ds \in C^{1,\gamma}$ then $\Lambda\int_0^t \boldsymbol{X}(s)ds \in C^{0,\gamma}$. Furthermore, from the local well-posedness theory, $\mathcal{R}(\boldsymbol{X}(t)) \in C^{\lfloor 2\gamma\rfloor, 2\gamma-\lfloor 2\gamma\rfloor}$. We can now define the finite difference,

$$\frac{\boldsymbol{X}(t+h)-\boldsymbol{X}(t)}{h} = \frac{1}{h}\int_t^{t+h}\Lambda\boldsymbol{X}(s)ds + \frac{1}{h}\int_t^{t+h}\mathcal{R}(\boldsymbol{X}(s))ds.$$

Thanks to Lipschitz continuity of $\mathcal{R}$ proved in Proposition 4.1.5, $\mathcal{R}(\boldsymbol{X})$ is continuous at t. Then,

$$\lim_{t\to 0}\frac{1}{h}\int_t^{t+h}\mathcal{R}(\boldsymbol{X}(s))ds = \mathcal{R}(\boldsymbol{X}(t)).$$

Likewise, $\lim_{t\to 0}\frac{1}{h}\int_t^{t+h}\Lambda\boldsymbol{X}(s)ds = \Lambda\boldsymbol{X}(t)$. Combining the limits yields (4.10) with $\partial_t\boldsymbol{X} \in C([0,T], C^{0,\gamma}(\mathbb{S}^1))$. $\square$

We also show that a strong solution satisfies the mild form of the equation. This fact will also be used in the stability analysis.

Lemma 4.1.8. *Let* $\boldsymbol{X}(t) \in C([0,t]; C^{1,\gamma}(\mathbb{S}^1)) \cap C^1([0,T]; C^{0,\gamma}(\mathbb{S}^1))$ *solve* (1.18) *with initial data* $\boldsymbol{X}_0$*, then* $\boldsymbol{X}(t)$ *satisfies* (1.19) *on the time interval* $[0,T]$.

Proof. If we fix $t > 0$ and define $\boldsymbol{Z}(s) = e^{(t-s)\Lambda}\boldsymbol{X}(s)$ then for any $0 \leq s < t$ we have

$$\begin{aligned}\partial_s \boldsymbol{Z}(s) &= e^{(t-s)\Lambda}\partial_s \boldsymbol{X}(s) - e^{(t-s)\Lambda}\Lambda \boldsymbol{X}(s)\\ &= e^{(t-s)\Lambda}\mathcal{R}(\boldsymbol{X}(s)),\end{aligned}$$

where $\partial_s \boldsymbol{X}(s), \Lambda \boldsymbol{X}(s) \in C^{0,\gamma}(\mathbb{S}^1)$. Integrating on $[0,t]$ yields

$$\boldsymbol{X}(t) - e^{t\Lambda}\boldsymbol{X}_0 = \int_0^t e^{(t-s)\Lambda}\mathcal{R}(\boldsymbol{X}(s))ds$$

and (1.19). □

We are now able to prove Theorem 1.2.3.

Proof. Theorem 1.2.3 We have already proved item (i) in Proposition 4.1.6 and item (iv) in Lemmas 4.1.7 and 4.1.8. We prove item (ii). Suppose we have two solutions $\boldsymbol{Y}$ and $\boldsymbol{Z}$ in $C([0,T]; C^{1,\gamma}(\mathbb{S}^1))$ with the same initial value $\boldsymbol{X}_0$. Define T_* to be:

$$T_* = \sup\{\tau : 0 \leq \tau \leq T,\ \boldsymbol{Y}(t) = \boldsymbol{Z}(t) \text{ for all } 0 \leq t \leq \tau\}$$

We show that $T_* = T$. Suppose otherwise. Then, $T_* < T$. First note that $\boldsymbol{Y}(T_*) = \boldsymbol{Z}(T_*) \in h^{1,\gamma}(\mathbb{S}^1)$. If $T_* = 0$, this is true by assumption. For $T_* > 0$, this follows from:

$$\|\boldsymbol{Y}(T_*)\|_{C^{1,\gamma'}} \leq \frac{C}{T_*^{\gamma'-\gamma}}\|\boldsymbol{X}_0\|_{C^{1,\gamma}} + \int_0^{T_*}\frac{C}{(T_*-s)^{\gamma'-\gamma}}\|\boldsymbol{Y}(s)\|_{C^{1,\gamma}}\, ds < \infty,$$

where $\gamma < \gamma' < 1$. Thus, $\boldsymbol{Y}(T_*) \in C^{1,\gamma'}(\mathbb{S}^1) \subset h^{1,\gamma}(\mathbb{S}^1)$. We also have $|\boldsymbol{Y}(T_*)|_* > 0$ by the definition of the mild solution. We may thus consider a mild solution to the Peskin problem starting at $t = T_*$ with initial value $\boldsymbol{X}_* = \boldsymbol{Y}(T_*) = \boldsymbol{Z}(T_*)$. By the contraction mapping argument of Proposition 4.1.6, there is a unique mild solution $\boldsymbol{W}(t)$ with initial data $\boldsymbol{X}_*$ for some time $0 \leq t \leq \widetilde{T}_* \leq T - T_*, \widetilde{T}_* > 0$. By the uniqueness of the fixed point of the contraction map, we must have $\boldsymbol{Y}(t+T_*) = \boldsymbol{W}(t) = \boldsymbol{Z}(t+T_*)$ for $0 \leq t \leq \widetilde{T}_*$. This is a contradiction.

We next prove item (iii). Let $\boldsymbol{X}(t)$ be a mild solution in $C([0,T];C^{1,\gamma}(\mathbb{S}^1))$ with initial data $\boldsymbol{X}_0$. Take some time $T_1>0$ and consider the set $\widetilde{\mathcal{B}}_{T_1}$:

$$\widetilde{\mathcal{B}}_{X(t_*),T_1}=\left\{\boldsymbol{X}\in C([0,T_1];C^{1,\gamma}(\mathbb{S}^1)):\|\boldsymbol{X}-\boldsymbol{X}(t_*)\|_{C([0,T];C^{1,\gamma})}\leq\frac{1}{2}\inf_{0\leq t\leq T}|\boldsymbol{X}(t)|_*\right\},$$

Set

$$m=\frac{1}{2}\inf_{0\leq t\leq T}|\boldsymbol{X}(t)|_*,\ M=m+\sup_{0\leq t\leq T}\|\boldsymbol{X}(t)\|_{C^{1,\gamma}}.$$

Consider the map $\mathcal{S}_{\boldsymbol{Y}_0}$ taking $\boldsymbol{Y}(t)$ to $\mathcal{S}(\boldsymbol{Y},t;\boldsymbol{Y}_0)$ (see (4.7)). We may estimate, in the same way as in (4.9),

$$\|S(\boldsymbol{X},t;\boldsymbol{X}_0)-\boldsymbol{Y}_0\|_{C^{1,\gamma}}\leq\left\|(e^{t\Lambda}-1)\boldsymbol{X}_0\right\|_{C^{1,\gamma}}+\|\boldsymbol{X}_0-\boldsymbol{Y}_0\|_{C^{1,\gamma}}+Ct^{\gamma}\frac{M^5}{m^4},\tag{4.13}$$

Thus, by taking $\|\boldsymbol{X}_0-\boldsymbol{Y}_0\|\,C^{1,\gamma}$ and T_1 small enough, we see that $\mathcal{S}_{\boldsymbol{Y}_0}$ maps $\mathcal{B}_{\boldsymbol{X}_0,T_1}$ to itself. That this is a contraction follows in the same way as Proposition 4.1.6. Thus, there is an $\epsilon>0$ and $T_1>0$ depending on $\boldsymbol{X}_0$ such that, for all initial data $\boldsymbol{Y}_0$ satisfying $\|\boldsymbol{Y}_0-\boldsymbol{X}_0\|_{C^{1,\gamma}}\leq\epsilon$, there is a mild solution $\boldsymbol{Y}(t)$ to the Peskin problem up to time T_1. We also see that

$$(\boldsymbol{X}-\boldsymbol{Y})(t)=e^{\Lambda t}(\boldsymbol{X}_0-\boldsymbol{Y}_0)+\int_0^t e^{\Lambda(t-s)}\left[\mathcal{R}(\boldsymbol{X})(s)-\mathcal{R}(\boldsymbol{Y})(s)\right]ds$$

for all $t\in[0,T_1]$. Taking the $C^{1,\gamma}$ norm of this equation and using Proposition 4.1.5 and semigroup estimate (3.14) gives

$$\|(\boldsymbol{X}-\boldsymbol{Y})(t)\|_{C^{1,\gamma}}\leq C\|\boldsymbol{X}_0-\boldsymbol{Y}_0\|_{C^{1,\gamma}}+C\frac{M^4}{m^4}\int_0^t(t-s)^{\gamma}\|(\boldsymbol{X}-\boldsymbol{Y})(s)\|_{C^{1,\gamma}}\,ds.$$

Making use of a generalized Grönwall's lemma from Lemma 7.0.3 of [21] gives

$$\|(\boldsymbol{X}-\boldsymbol{Y})(t)\|_{C^{1,\gamma}}\leq C\|\boldsymbol{X}_0-\boldsymbol{Y}_0\|_{C^{1,\gamma}}\leq C\epsilon.$$

Since $t\in[0,T]$ was arbitrary, taking the supremum over all $t\in[0,T_1]$ this shows that we have continuity in $C([0,T_1];C^{1,\gamma}(\mathbb{S}^1))$. Selecting $t_*=T_1$ and repeating this argument with initial data $\boldsymbol{Y}_1$ close to $\boldsymbol{X}(T_1)$, by possibly reducing the value of ϵ, we can extend our result up to $t=T_1$. This process can be repeated until we reach $t=T$. $\square$

4.2 Smoothness and Equivalence of Solutions

In this section we first prove higher regularity of our solution in space and time in Subsection 4.2.1. Once the regularity of the boundary integral formulation of our problem has been established, we show that our solution is equivalent to the other formulations of the Peskin problem in subsection 4.2.2. For $n \in \mathbb{N}$ and $\alpha > 0$, we occasionally write $C^{n+\alpha}$ to mean $C^{n+\lfloor\alpha\rfloor,\alpha-\lfloor\alpha\rfloor}$ for convenience.

4.2.1 Improved Regularity of Solutions

Given that the symbol of our leading order operator is parabolic, it is natural to ask whether the contours will become smooth as time evolves. In order to establish higher regularity, our method involves converting spatial derivatives in θ on the integral operator into derivatives in θ', similar to a method developed in [11]. Our goal is to show that the nonlinear remainder terms carry the regularity of $\boldsymbol{X}$, and improved regularity on $\boldsymbol{X}$ follows from an application of semigroup estimate (3.14).

We introduce the notation

$$\boldsymbol{\chi}_j := \Delta\partial_\theta^j \boldsymbol{X} = \partial_\theta^j \boldsymbol{X} - \partial_{\theta'}^j \boldsymbol{X}',\ \boldsymbol{\chi}_j = (\chi_{j,1}, \chi_{j,2}),$$

for $\boldsymbol{X} \in C^{j,\gamma}(\mathbb{S}^1)$ and $|\boldsymbol{X}|_* > 0$ and the following sets

$$\mathcal{S}_k = \left\{ \sum_{l=0}^{M} a_l g_l : a_l \in \mathbb{R}, g_l \in \mathcal{S}_k^1, M \in \mathbb{N} \right\},$$

$$\mathcal{S}_k^1 = \left\{ \prod_{l=0}^{k} \left(\frac{\chi_{l,1}}{|\boldsymbol{\chi}_0|}\right)^{\alpha_l} \left(\frac{\chi_{l,2}}{|\boldsymbol{\chi}_0|}\right)^{\beta_l}, \alpha_l, \beta_l \in \mathbb{N} \cup \{0\},\ \sum_{l=0}^{k} l(\alpha_l + \beta_l) = k \right\}.$$

In this section, a term in $\mathcal{S}_k^1$ will be called a k-monomial. Note that the sum:

$$\sum_{l=0}^{k} l(\alpha_l + \beta_l) \tag{4.14}$$

is the total number of derivatives in one k-monomial. One may thus say that a k-monomial is a monomial that has exactly k derivatives. Since all α_l, β_l are positive, this implies that the largest l for which α_l, β_l can be non-zero is $l = k$. Also, note that a k-monomial satisfies the assumptions for the function g of Lemma 2.2.3 and 2.2.4 so long as $\boldsymbol{X} \in C^{k+1,\gamma}(\mathbb{S}^1),\ \gamma \in (0,1)$. The number N that appears in the estimates of Lemmas 2.2.3 and 2.2.4 are given by:

$$N = \sum_{l=1}^{k} (\alpha_k + \beta_k) \leq k. \tag{4.15}$$

The relevance of the class of k-monomials and their linear combinations to the problem at hand is contained in the following lemma.

Lemma 4.2.1. *If $k \geq 0$, $f \in \mathcal{S}_k$, and $\boldsymbol{X} \in C^{k+1,\gamma}(\mathbb{S}^1)$ for some $\gamma \in (0,1)$ with $|\boldsymbol{X}|_* > 0$ then*

$$\partial_\theta f + \partial_{\theta'} f \in \mathcal{S}_{k+1}. \tag{4.16}$$

Proof. It is clear that we have only to prove the assertion for one k-monomial $f \in S_k^1$. Let us adopt the notation of Lemma 2.2.3. We set:

$$f = \prod_{l=0}^{k} \phi_l^{\alpha_l} \psi_l^{\beta_l},\ \phi_l = \frac{\chi_{l,1}}{|\boldsymbol{\chi}_0|},\ \psi_l = \frac{\chi_{l,2}}{|\boldsymbol{\chi}_0|}.$$

Let us compute $\partial_\theta f + \partial_{\theta'} f$. We have:

$$\begin{aligned}
\partial_\theta f + \partial_{\theta'} f = \sum_{l=0}^{k} (\alpha_l A_l + \beta_l B_l)&, \\
A_l = (\partial_\theta \phi_l + \partial_{\theta'} \phi_l) \phi_l^{\alpha_l - 1} \psi_l^{\beta_l} \prod_{i \neq l} \phi_i^{\alpha_i} \psi_i^{\beta_i}&, \\
B_l = (\partial_\theta \psi_l + \partial_{\theta'} \psi_l) \phi_l^{\alpha_l} \psi_l^{\beta_l - 1} \prod_{i \neq l} \phi_i^{\alpha_i} \psi_i^{\beta_i}&.
\end{aligned}$$

We may compute:

$$\partial_\theta \phi_l + \partial_{\theta'} \phi_l = \frac{\chi_{l+1,1}}{|\boldsymbol{\chi}_0|} - \frac{\chi_{l,1} \boldsymbol{\chi}_0 \cdot \boldsymbol{\chi}_1}{|\boldsymbol{\chi}_0|^3}.$$

It is clear that each A_l gives rise to a linear combination of monomials and that each monomial has $k+1$ total derivatives. Therefore, each monomial in A_l is a $(k+1)$-monomial belonging to $\mathcal{S}_{k+1}^1$. The same considerations apply for B_l. This concludes the proof. $\square$

The next lemma provides an explicit representation of high order derivatives on the nonlinear remainder $\mathcal{R}(\boldsymbol{X})$.

Lemma 4.2.2. *Assume $\boldsymbol{X} \in C^{k+1,\alpha}(\mathbb{S}^1)$ with $k \geq 1$ and $\alpha \in (0,1)$, and assume $|\boldsymbol{X}|_* > 0$. Then*

$$\partial_\theta^k \mathcal{R}\left(\boldsymbol{X}\right) = \sum_{j=1}^{k} \int_{\mathbb{S}^1} \partial_{\theta'} P_{k+1-j}^k \partial_{\theta'}^j \boldsymbol{X}' d\theta' + F_C[\partial_\theta^{k+1}\boldsymbol{X}] + F_T[\partial_\theta^{k+1}\boldsymbol{X}], \tag{4.17}$$

where $P_{k+1-j}^k \in \mathcal{S}_{k+1-j}$ for $j \in \{1,\ldots,k\}$.

Proof. We will prove (4.17) inductively using the structure of the nonlinear remainder and the form of the kernels in the class, $\mathcal{S}_j$. First recall

$$\mathcal{R}(\boldsymbol{X}) = F_C[\partial_\theta \boldsymbol{X}] + F_T[\partial_\theta \boldsymbol{X}]$$

where

$$\begin{aligned} F_C[\boldsymbol{u}] &= \frac{1}{4\pi}\int_{\mathbb{S}^1} (\partial_{\theta'} M_0)\boldsymbol{u}' d\theta' \\ M_0(\theta,\theta') &= \log\left(\frac{|\Delta \boldsymbol{X}|}{2\left|\sin((\theta-\theta')/2)\right|}\right) = \log\left(\frac{|\boldsymbol{\chi}_0|}{2\left|\sin((\theta-\theta')/2)\right|}\right), \end{aligned}$$

and

$$\begin{aligned} F_T[\boldsymbol{u}] &= \frac{1}{4\pi}\int_{\mathbb{S}^1} (\partial_\theta' N_0)\boldsymbol{u}' d\theta', \\ N_0(\theta,\theta') &= -\left(\frac{\Delta \boldsymbol{X} \otimes \Delta \boldsymbol{X}}{|\Delta \boldsymbol{X}|^2}\right) = -\left(\frac{\boldsymbol{\chi}_0 \otimes \boldsymbol{\chi}_0}{|\boldsymbol{\chi}_0|^2}\right) \in \mathcal{S}_0. \end{aligned}$$

We first note that we can convert derivatives in θ into θ' derivatives so long as $\boldsymbol{u} \in C^1(\mathbb{S}^1)$ and $\boldsymbol{X} \in C^2(\mathbb{S}^1)$ as follows. Using (2.73) and (2.69), we have:

$$\begin{aligned} \partial_\theta F_C[\boldsymbol{u}] &= \frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'}\partial_\theta M_0(\boldsymbol{u}' - \boldsymbol{u}) d\theta' \\ &= \frac{1}{4\pi}\int_{\mathbb{S}^1} \left(-\partial_{\theta'}^2 M_0 + \partial_\theta' \left(\partial_\theta M_0 + \partial_{\theta'} M_0\right)(\boldsymbol{u}' - \boldsymbol{u})\right) d\theta' \\ &= \frac{1}{4\pi}\int_{\mathbb{S}^1} \left(\partial_{\theta'} M_0(\partial_{\theta'}\boldsymbol{u}') + \partial_{\theta'}\left((\partial_\theta + \partial_{\theta'})M_0\right)(\boldsymbol{u}' - \boldsymbol{u})\right) d\theta' \end{aligned} \tag{4.18}$$

Let

$$M_1 = \partial_\theta M_0 + \partial_{\theta'} M_0 = \frac{\boldsymbol{\chi}_0 \cdot \boldsymbol{\chi}_1}{|\boldsymbol{\chi}_0|^2} \in \mathcal{S}_1.$$

From (4.18), we have:

$$\partial_\theta F_C[\boldsymbol{u}] = F_C[\partial_\theta \boldsymbol{u}] + \frac{1}{4\pi}\int_{\mathbb{S}^1} (\partial_\theta' M_1)\boldsymbol{u}' d\theta' \tag{4.19}$$

where we used the fact that M_1 is continuous so that the integral $\partial'_\theta M_1 \boldsymbol{u}$ vanishes.

In much the same way, we have:

$$\partial_\theta F_T[\boldsymbol{u}] = F_T[\partial_\theta \boldsymbol{u}] + \frac{1}{4\pi}\int_{\mathbb{S}^1} (\partial'_\theta N_1)\boldsymbol{u}' d\theta' \tag{4.20}$$

where, by lemma 4.2.1,

$$N_1 = \partial_\theta N_0 + \partial_{\theta'} N_0 \in \mathcal{S}_1.$$

Combining (4.19) and (4.20), we have:

$$\begin{gathered}\partial_\theta F_C[\boldsymbol{u}] + \partial_\theta F_T[\boldsymbol{u}] = \int_{\mathbb{S}^1} \partial_{\theta'} P_1^1 \boldsymbol{u}' d\theta' + F_C[\partial_\theta \boldsymbol{u}] + F_T[\partial_\theta \boldsymbol{u}], \\ P_1^1 := \frac{1}{4\pi}(M_1 + N_1) \in \mathcal{S}_1\end{gathered} \tag{4.21}$$

In particular, we can write,

$$\partial_\theta \mathcal{R}(\boldsymbol{X}) = \int_{\mathbb{S}^1} \partial_{\theta'} P_1^1 \partial_{\theta'} \boldsymbol{X}' d\theta' + F_C\left[\partial_\theta^2 \boldsymbol{X}\right] + F_T\left[\partial_\theta^2 \boldsymbol{X}\right] \tag{4.22}$$

which satisfies (4.17) with $k = 1$.

We proceed by mathematical induction. Suppose the statement of the Proposition is true for some $k \geq 1$. We show that it is true for $k+1$. Assume $\boldsymbol{X} \in C^{k+2,\alpha}(\mathbb{S}^1)$. Since $\boldsymbol{X} \in C^{k+2,\alpha}(\mathbb{S}^1) \subset C^{k+1,\alpha}(\mathbb{S}^1)$, (4.17) is true by the induction hypothesis. If we differentiate (4.17) in θ and write out the resulting nonlinear kernel, we find

$$\begin{aligned}\partial_\theta^{k+1}\mathcal{R}(\boldsymbol{X}) = \sum_{j=1}^{k} \int_{\mathbb{S}^1} \partial_{\theta'}\partial_\theta \left(P_{k+1-j}^k\right)\left(\partial_{\theta'}^j \boldsymbol{X}' - \partial_\theta^j \boldsymbol{X}\right) d\theta' \\ + \partial_\theta F_C[\partial_\theta^{k+1}\boldsymbol{X}] + \partial_\theta F_T[\partial_\theta^{k+1}\boldsymbol{X}].\end{aligned} \tag{4.23}$$

A rigorous justification of this calculation will proceed in the same way as we obtained (2.73) in the proof of Proposition 4.1.1. We omit this proof. Since the contour $\boldsymbol{X}$ is smooth enough, we can use (4.16) to move derivatives off of the kernels. In particular we write:

$$\partial_\theta P_{k+1-j}^k = -\partial_{\theta'} P_{k+1-j}^k + \widetilde{P}_{k+2-j}^k$$

for some new multipliers $\widetilde{P}_{k+2-j}^k \in \mathcal{S}_{k+2-j}$. Furthermore, from (4.21),

$$\partial_\theta F_C[\partial_\theta^{k+1}\boldsymbol{X}] + \partial_\theta F_T[\partial_\theta^{k+1}\boldsymbol{X}] = \int_{\mathbb{S}^1} \partial_{\theta'} P_1^1 \partial_{\theta'}^{k+1} \boldsymbol{X}' d\theta' + F_C[\partial_\theta^{k+2}\boldsymbol{X}] + F_T[\partial_\theta^{k+2}\boldsymbol{X}].$$

We can now rewrite the terms of (4.23) as follows:

$$
\begin{aligned}
\partial_\theta^{k+1}\mathcal{R}(\boldsymbol{X}) &= \sum_{j=1}^{k}\int_{\mathbb{S}^1}\partial_{\theta'}\left(-\partial_{\theta'}P^k_{k+1-j}+\widetilde{P}^k_{k+2-j}\right)\left(\partial^j_{\theta'}\boldsymbol{X}'-\partial^j_\theta\boldsymbol{X}\right)d\theta' \\
&\quad+\int_{\mathbb{S}^1}\partial_{\theta'}P^1_1\partial^{k+1}_{\theta'}\boldsymbol{X}'d\theta'+F_C[\partial_\theta^{k+2}\boldsymbol{X}]+F_T[\partial_\theta^{k+2}\boldsymbol{X}] \\
&=\sum_{j=1}^{k}\left(\int_{\mathbb{S}^1}\partial_{\theta'}P^k_{k+1-j}\partial^{j+1}_{\theta'}\boldsymbol{X}'d\theta'+\int_{\mathbb{S}^1}\partial_{\theta'}\widetilde{P}^k_{k+2-j}\partial^j_{\theta'}\boldsymbol{X}'d\theta'\right) \\
&\quad+\int_{\mathbb{S}^1}\partial_{\theta'}P^1_1\partial^{k+1}_{\theta'}\boldsymbol{X}'d\theta'+F_C[\partial_\theta^{k+2}\boldsymbol{X}]+F_T[\partial_\theta^{k+2}\boldsymbol{X}] \\
&=\sum_{j=1}^{k+1}\int_{\mathbb{S}^1}\partial_{\theta'}P^{k+1}_{k+2-j}\partial^j_{\theta'}\boldsymbol{X}'d\theta'+F_C[\partial_\theta^{k+2}\boldsymbol{X}]+F_T[\partial_\theta^{k+2}\boldsymbol{X}],
\end{aligned}
$$

where we defined

$$
P^{k+1}_{k+2-j}=\begin{cases}\widetilde{P}^k_{k+1} & \text{if } j=1\\ P^k_{k+2-j}+\widetilde{P}^k_{k+2-j} & \text{if } 2\le j\le k\\ P^k_1+P^1_1 & \text{if } j=k+1.\end{cases}
$$

It may be easily checked that $P^{k+1}_{k+2-j}\in S_{k+2-j}$ for $1\le j\le k+1$. □

For the next step, we use our calculus lemmas to show that the remainder terms satisfy the following smoothing estimate.

Lemma 4.2.3. *Assume $\boldsymbol{X}\in C^{n,\alpha}(\mathbb{S}^1)$ with $n\ge 2$ where $\alpha\in(1/2,1)$, and assume $|\boldsymbol{X}|_*>0$. Then*

$$
\|\mathcal{R}(\boldsymbol{X})\|_{C^{n,2\alpha-1}}\le C\left(\frac{\|\boldsymbol{X}\|_{C^{n,\alpha}}}{|\boldsymbol{X}|_*}\right)^{n+2}\|\boldsymbol{X}\|_{C^{n,\alpha}}\,, \tag{4.24}
$$

where the constant C depends only on n and α.

As the reader will see from the proof, the above bound is suboptimal. It will, however, be sufficient for our purposes.

Proof of Lemma 4.2.3. In order to prove lemma 4.2.3, we express the terms of (4.17) in three parts,

$$
\partial_\theta^{n-1}\mathcal{R}(\boldsymbol{X})=G_{n-1}+F_C[\partial^n_\theta\boldsymbol{X}]+F_T[\partial^n_\theta\boldsymbol{X}],
$$

where

$$G_{n-1}=\sum_{j=1}^{n-1}G_j^{n-1}:=\sum_{j=1}^{n-1}\int_{\mathbb{S}^1}\partial_{\theta'}P_{n-j}^{n-1}\partial_{\theta'}^j\boldsymbol{X}'d\theta'.$$

From Propositions 4.1.1 and 4.1.2 we have the bounds

$$\left\|\partial_\theta F_C[\partial_\theta^{n-1}\boldsymbol{X}]+\partial_\theta F_T[\partial_\theta^{n-1}\boldsymbol{X}]\right\|_{C^{0,2\alpha-1}}\leq C\frac{\|\boldsymbol{X}\|_{C^{1,\alpha}}^3}{|\boldsymbol{X}|_*^3}\|\boldsymbol{X}\|_{C^{n,\alpha}}. \tag{4.25}$$

We must now show that $\partial_\theta G_j^{n-1}$ are in $C^{0,2\alpha-1}$. The important point here is that $P_{n-j}^{n-1}\in\mathcal{S}_{n-j}$, and thus Lemmas 2.2.3 and 2.2.4 are applicable. Suppose P_{n-j}^{n-1} can be written as:

$$P_{n-j}^{n-1}=\sum_{\ell=1}^{M}a_\ell g_\ell,\ g_\ell\in\mathcal{S}_{n-j}^1.$$

Any of the terms g_ℓ has the form:

$$g_\ell=\prod_{l=0}^{n-j}\left(\frac{\chi_{l,1}}{|\boldsymbol{\chi}_0|}\right)^{\alpha_l}\left(\frac{\chi_{l,2}}{|\boldsymbol{\chi}_0|}\right)^{\beta_l},\quad \sum_{l=1}^{n-j}l(\alpha_l+\beta_l)=n-j.$$

Then, using a procedure that is exactly the same as the proof of Proposition 4.1.2 (or Proposition 4.1.1), we have:

$$\left\|\partial_\theta\int_{\mathbb{S}^1}\partial_{\theta'}g_\ell\partial_{\theta'}^j\boldsymbol{X}'d\theta'\right\|_{C^{0,2\alpha-1}}=C\frac{\|\boldsymbol{X}\|_{C^{1,\alpha}}^3\prod_{l=1}^{n-j}\left\|\partial_\theta^l\boldsymbol{X}\right\|_{C^{1,\alpha}}^{\alpha_l+\beta_l}}{|\boldsymbol{X}|_*^{N+3}}\left\|\partial_\theta^j\boldsymbol{X}\right\|_{C^{0,\alpha}}$$

where N is given as in (4.15). A rather crude over-estimation yields:

$$\left\|\partial_\theta G_j^{n-1}\right\|_{C^{0,2\alpha-1}}\leq C\left(\frac{\|\boldsymbol{X}\|_{C^{n-j+1,\alpha}}}{|\boldsymbol{X}|_*}\right)^{n-j+3}\|\boldsymbol{X}\|_{C^{j,\alpha}}. \tag{4.26}$$

Combining (4.25) and (4.26), we obtain the following estimate:

$$\|\partial_\theta^n\mathcal{R}(\boldsymbol{X})\|_{C^{0,2\alpha-1}}\leq C\left(\frac{\|\boldsymbol{X}\|_{C^{n,\alpha}}}{|\boldsymbol{X}|_*}\right)^{n+2}\|\boldsymbol{X}\|_{C^{n,\alpha}}.$$

Recall from Proposition 4.1.1 and 4.1.2 that

$$\|\mathcal{R}(\boldsymbol{X})\|_{C^{1,2\alpha-1}}\leq C\left(\frac{\|\boldsymbol{X}\|_{C^{1,\alpha}}}{|\boldsymbol{X}|_*}\right)^3\|\boldsymbol{X}\|_{C^{1,\alpha}}.$$

We thus have the desired estimate. $\square$

Finally, using the above proposition, we have the following proposition.

Proposition 4.2.4. *Suppose $\boldsymbol{X}(t)$ is a mild solution to the Peskin problem in the interval $[0,T]$ with initial $\boldsymbol{X}_0 \in h^{1,\gamma}(\mathbb{S}^1)$. For any $n \in \mathbb{N}$, $\gamma \in (0,1)$, $\boldsymbol{X}(t)$ is in $C([\epsilon,T];C^{n,\alpha}(\mathbb{S}^1))$ for any $\epsilon \in (0,T)$ and any $\alpha \in (0,1)$.*

Proof. 1. To prove higher regularity, we will be estimating a differentiated form of the mild solution, expressed in a more general form,

$$\boldsymbol{Y}(t) = e^{(t-t_0)\Lambda}\boldsymbol{Y}(t_0) + \int_{t_0}^{t} e^{(t-s)\Lambda} F(\boldsymbol{Y}(s))ds. \tag{4.27}$$

If we assume that $\boldsymbol{Y}(t_0) \in C^{\beta}(\mathbb{S}^1)$ and $F(\boldsymbol{Y}(s)) \in C([t_0,T];C^{\alpha}(\mathbb{S}^1))$ for all $t_0 \leq s \leq t$ then the semigroup estimate (3.14), implies that for any $\alpha \geq 0$ and $\delta \in (0,1]$ with $1+\alpha-\delta$ non-integer valued,

$$\|\boldsymbol{Y}(t)\|_{C^{1+\alpha-\delta}} \leq \frac{C}{(t-t_0)^{1+\alpha-\beta-\delta}} \|\boldsymbol{Y}(t_0)\|_{C^{\beta}} + C \sup_{t_0 \leq s \leq t} \|F(\boldsymbol{Y}(s))\|_{C^{\alpha}}. \tag{4.28}$$

This follows from (3.14) and

$$\begin{aligned}\|\boldsymbol{Y}(t)\|_{C^{1+\alpha-\delta}} &\leq \left\|e^{t\Lambda}\boldsymbol{Y}(t_0)\right\|_{C^{1+\alpha-\delta}} + \int_{t_0}^{t} \left\|e^{(t-s)\Lambda}F(\boldsymbol{Y}(s))\right\|_{C^{1+\alpha-\delta}} \\ &\leq \frac{C}{(t-t_0)^{1+\alpha-\delta-\beta}} \|\boldsymbol{Y}(t_0)\|_{C^{\beta}} + \int_{t_0}^{t} (t-s)^{\delta-1} \|F(\boldsymbol{Y}(s))\|_{C^{\alpha}} ds.\end{aligned}$$

We will be moving t_0 away from zero during the iteration procedure.

2. Let $\boldsymbol{X} \in C([0,T];C^{1,\gamma}(\mathbb{S}^1))$ be the mild solution generated in Theorem 1.2.3. We first claim that for any $\epsilon_2 \in (0,\epsilon)$ our solution $\boldsymbol{X} \in L^{\infty}([\epsilon_2,T];C^{2,\alpha}(\mathbb{S}^1))$, for all $0<\alpha<1$. Here, $L^{\infty}([\epsilon,T];C^{k,\beta}(\mathbb{S}^1))$ refers to the set of functions of t with values in $C^{k,\beta}(\mathbb{S}^1)$ with bounded $C^{k,\beta}(\mathbb{S}^1)$ norm for $\epsilon \leq 0 \leq T$. To show this we define an iteration $\gamma_{(1,k)}$ starting with $\gamma_{(1,0)} = \gamma$ and updates $\gamma_{(1,j+1)} = \frac{3}{2}\gamma_{(1,j)}$. We stop the iteration once $1 < \gamma_{(1,k+1)} < 2$, i.e. $k = \lfloor \frac{\ln 2/\gamma}{\ln 3/2} \rfloor$. Without loss of generality, we may assume that none of these $\gamma_{1,j}$'s fall on an integer value. Subdividing $0 < \epsilon_{(1,1)} < \epsilon_{(1,2)} < \cdots < \epsilon_{(1,k)} < \epsilon_2$, we use (4.28) and if $\boldsymbol{X} \in L^{\infty}([\epsilon_{(1,j)},T];C^{1+\gamma_{(1,j)}}(\mathbb{S}^1))$ then

$$\|\boldsymbol{X}\|_{C^{1+\frac{3}{2}\gamma_{(1,j)}}} \leq C\frac{1}{(t-\epsilon_{(1,j)})^{\frac{\gamma_{(1,j)}}{2}}} \left\|\boldsymbol{X}(\epsilon_{(1,j)})\right\|_{C^{1+\gamma_{(1,j)}}} + C \sup_{\epsilon_{1,j} \leq s \leq t} \|\mathcal{R}(\boldsymbol{X}(s))\|_{C^{\frac{7}{4}\gamma_{(1,j)}}},$$

where we have chosen $\alpha = \frac{7}{4}\gamma_{(1,j)}$ and $\delta = \frac{1}{4}\gamma_{(1,j)}$ in (4.28). Since

$$\|\mathcal{R}(\boldsymbol{X}(s))\|_{C^{\frac{7}{4}\gamma_{(1,j)}}} \leq C(\|\boldsymbol{X}\|_{C^{1+\frac{7}{8}\gamma_{(1,j)}}}, |\boldsymbol{X}|_*) \leq C(\|\boldsymbol{X}\|_{C^{1+\gamma_{(1,j)}}}, |\boldsymbol{X}|_*),$$

by Propositions 4.1.1, 4.1.2 and the induction hypothesis, then
$\boldsymbol{X} \in L^\infty([\epsilon_{(1,j+1)}, T]; C^{1+\gamma_{(1,j+1)}}(\mathbb{S}^1))$.

Therefore, we have $\boldsymbol{X} \in L^\infty([\epsilon_{(1,k)}, T]; C^{1+\gamma_{(1,k+1)}}(\mathbb{S}^1))$ for some $\gamma_{(1,k+1)} \in (1,2)$, which implies that $\boldsymbol{X} \in L^\infty([\epsilon_{(1,k)}, T]; C^{2,\tilde{\gamma}}(\mathbb{S}^1))$ for some $\tilde{\gamma} \in (0,1)$. Finally we choose $\alpha = 2 - \delta$ for any small $\delta > 0$, then (3.14) implies

$$\|\boldsymbol{X}\|_{C^{2,1-2\delta}} \leq C\frac{1}{(t-\epsilon_{(1,k)})^{1-2\delta-\tilde{\gamma}}}\left\|\boldsymbol{X}(\epsilon_{(1,j)})\right\|_{C^{2+\tilde{\gamma}}} + C\sup_{\epsilon_{(1,k)} \leq s \leq t}\|\mathcal{R}(\boldsymbol{X}(s))\|_{C^{1,1-\delta}},$$

and since $\|\mathcal{R}(\boldsymbol{X}(s))\|_{C^{1,1-\delta}} \leq C(\|\boldsymbol{X}(s)\|_{C^{1,1-\frac{\delta}{2}}}, |\boldsymbol{X}|_*) \leq C(\|\boldsymbol{X}(s)\|_{C^{2,\tilde{\gamma}}}, |\boldsymbol{X}|_*)$, by Propositions 4.1.1, 4.1.2, and since $\delta > 0$ is arbitrarily small, we achieve regularity for the full range of Hölder norms.

3. We now show that $\boldsymbol{X} \in L^\infty([\epsilon, T], C^{n,\alpha}(\mathbb{S}^1))$ for any $\alpha \in (0,1)$. To show this we subdivide $0 < \epsilon_2 < \epsilon_3 < \ldots < \epsilon_n = \epsilon$, and we suppose that $\boldsymbol{X} \in L^\infty([\epsilon_k, T], C^{k,\alpha}(\mathbb{S}^1))$ for some $k \geq 2$ and all $\alpha \in (0,1)$. By lemma 4.2.3 we find that $\partial_\theta^k \mathcal{R}(\boldsymbol{X}) \in C^{0,\tilde{\alpha}}(\mathbb{S}^1)$ for any $\tilde{\alpha} \in (0,1)$. We can then use estimate (4.28) and find that

$$\left\|\partial_\theta^k \boldsymbol{X}(t)\right\|_{C^{1,\tilde{\alpha}-\delta}} \leq \frac{C}{(t-t_0)^{1+\tilde{\alpha}-2\delta}}\left\|\partial_\theta^k \boldsymbol{X}(\epsilon_k)\right\|_{C^{0,\delta}} + \sup_{\epsilon_k \leq s \leq t}\left\|\partial_\theta^k \mathcal{R}(\boldsymbol{X}(s))\right\|_{C^{0,\tilde{\alpha}}},$$

and since $\tilde{\alpha}$ can be made arbitrary in $(0,1)$, then $\boldsymbol{X} \in L^\infty([\epsilon_{k+1}, T]; C^{k+1,1-2\delta}(\mathbb{S}^1))$ by suitable choices of $\alpha, \tilde{\alpha}, \delta$, and so $\boldsymbol{X} \in L^\infty([\epsilon_{k+1}, T]; C^{k+1,\alpha}(\mathbb{S}^1))$ for any $\alpha \in (0,1)$. This process can be repeated $n-1$ times to conclude that $\boldsymbol{X} \in L^\infty([\epsilon_n, T]; C^{n,\alpha}(\mathbb{S}^1))$ for any $\alpha \in (0,1)$. In particular we see that $\boldsymbol{X} \in L^\infty([\epsilon, T]; C^{n,\alpha}(\mathbb{S}^1))$ for any $\alpha \in (0,1)$.

4. We must show that $\boldsymbol{X} \in C([\epsilon, T]; C^{n,\alpha}(\mathbb{S}^1))$. Using interpolation on Hölder norms (see, for example, Chapter 1 of [21]), we have, for any $\epsilon \leq t_1 \leq t_2 \leq T$,

$$\begin{aligned}
\|\boldsymbol{X}(t_1) - \boldsymbol{X}(t_2)\|_{C^{n,\alpha}} &\leq C\,\|\boldsymbol{X}(t_1) - \boldsymbol{X}(t_2)\|_{C^{k,\beta}}^{1-\delta}\,\|\boldsymbol{X}(t_1) - \boldsymbol{X}(t_2)\|_{C^{1,\gamma}}^{\delta} \\
&\leq C\left(\|\boldsymbol{X}(t_1)\|_{C^{k,\beta}} + \|\boldsymbol{X}(t_2)\|_{C^{k,\beta}}\right)^{1-\delta}\|\boldsymbol{X}(t_1) - \boldsymbol{X}(t_2)\|_{C^{1,\gamma}}^{\delta} \\
&\quad \text{where } n + \alpha = \delta(1+\gamma) + (1-\delta)(k+\beta), \delta > 0.
\end{aligned}$$

Note that $\boldsymbol{X}(t) \in L^\infty([\epsilon, T]; C^{k,\beta}(\mathbb{S}^1)$ for any $k \in \mathbb{N}$ and $\beta \in (0,1)$. The above inequality can therefore be made to hold for any $n \in \mathbb{N}$ and $\alpha \in (0,1)$ by choosing $k + \beta$ large enough. We thus see that $\boldsymbol{X} \in C([\epsilon, T]; C^{n,\alpha}(\mathbb{S}^1))$ for any $n \in \mathbb{N}$, $\alpha \in (0,1)$. $\square$

Proof of Theorem 1.2.4. We show that the solution is in $C^1([\epsilon, T]; C^{k,\beta}(\mathbb{S}^1))$ for $\epsilon > 0$ for any n and $\alpha \in (0, 1)$. Recall from Theorem 1.2.3 that the mild solution $\boldsymbol{X}(t)$ is a strong solution and thus,

$$\partial_t \boldsymbol{X}(t) \in C^0([\epsilon, T]; C^{0,\gamma}(\mathbb{S}^1)). \tag{4.29}$$

Since $\boldsymbol{X}(t)$ is a strong solution, it satisfies:

$$\partial_t \boldsymbol{X} = \Lambda \boldsymbol{X} + \mathcal{R}(\boldsymbol{X}).$$

By the previous Proposition, $\boldsymbol{X} \in C([\epsilon, T]; C^{n+1,\alpha}(\mathbb{S}^1))$ for any $n \in \mathbb{N}, \alpha \in (0, 1)$. Thus, $\Lambda \boldsymbol{X} \in C([\epsilon, T]; C^{n,\alpha}(\mathbb{S}^1))$. Furthermore, by Lemma 4.2.3,
$\mathcal{R}(\boldsymbol{X}) \in L^\infty([\epsilon, T]; C^{n+1,2\alpha-1}(\mathbb{S}^1))$ so long as $\alpha \in (1/2, 1)$. Thus, we see that $\partial_t \boldsymbol{X} \in L^\infty([\epsilon, T]; C^{n,\alpha}(\mathbb{S}^1))$ for any $n \in \mathbb{N}$ and $\alpha \in (0, 1)$. That $\partial_t \boldsymbol{X} \in C([\epsilon, T]; C^{n,\alpha}(\mathbb{S}^1))$ now follows from the same argument as in Step 4 of the proof of Proposition 4.2.4, interpolating (4.29) with the bound $\partial_t \boldsymbol{X} \in L^\infty([\epsilon, T]; C^{k,\beta}(\mathbb{S}^1))$ for large enough $k + \beta$. $\square$

Note that Theorem 1.2.4 was proved using continuity of $\boldsymbol{X}(t)$ in time at low order Hölder norms and bounds for $\boldsymbol{X}(t)$ in higher order Hölder norms. The former is the result of Lipschitz continuity of $\mathcal{R}$ in $C^{1,\gamma}(\mathbb{S}^1)$ given in Proposition 4.1.5 and the latter of bounds on $\mathcal{R}$ in higher order Hölder norms given in lemma 4.2.3. To obtain higher order regularity in time, it would thus suffice to establish continuity of derivatives of $\mathcal{R}$ in $C^{1,\gamma}(\mathbb{S}^1)$ and bounds on such derivatives in higher order Hölder norms. One should be able to obtain the former by a generalization of the proof in proposition 4.1.5. Indeed, we will establish Lipschitz continuity of the derivative of $\mathcal{R}$ in proposition 4.3.3, which will be used in the study of stability of steady states. Bounds on the derivative of $\mathcal{R}$ in higher order Hölder norms should follow by arguments similar to those that led to lemma 4.2.3. We will not pursue this here.

4.2.2 Classical Solutions and the Equivalence of Formulations

In this subsection, we discuss classical solutions and the equivalence of formulations. Given the three formulations we have of the Peskin problem introduced in the introduction, we will introduce three notions of classical solutions corresponding to each formulation.

In the statement below, we let $C^k(\Omega_{\mathrm{i,e}})$ denote C^k functions in $\Omega_{\mathrm{i,e}}$ respectively and $C^k(\overline{\Omega}_{\mathrm{i,e}})$ denote the C^k functions in $\Omega_{\mathrm{i,e}}$ whose derivatives up to order k are uniformly continuous (so that limiting values of the derivatives are well-defined at the interface Γ).

Definition 4.2.5 (Classical jump and IB solutions)**.** *Let* $\boldsymbol{X}(t) \in C((0,T]; C^2(\mathbb{S}^1)) \cap C^1((0,T]; C(\mathbb{S}^1))$ *and* $\boldsymbol{X}_0 \in C^{1,\gamma}(\mathbb{S}^1),\ 0 < \gamma < 1$ *and* $|\boldsymbol{X}_0|_* > 0$. *The function* $\boldsymbol{X}(t)$ *is a* classical jump solution *with initial data* $\boldsymbol{X}_0$ *if:*

(i) *For each* t *satisfying* $0 < t \le T$*, there exist functions* $\boldsymbol{u}(\boldsymbol{x}, t)$ *and* $p(\boldsymbol{x}, t)$ *satisfying equations* (1.2)-(1.5)*,* (1.6) *and* (1.8)*. The velocity field* $\boldsymbol{u}$ *restricted to* $\Omega_{\mathrm{i,e}}$ *is in* $C^2(\Omega_{\mathrm{i,e}}) \cap C^1(\overline{\Omega}_{\mathrm{i,e}})$ *respectively and pressure* p *restricted to* $\Omega_{\mathrm{i,e}}$ *is in* $C^1(\Omega_{\mathrm{i,e}}) \cap C(\overline{\Omega}_{\mathrm{i,e}})$ *respectively.*

(ii) $\boldsymbol{X}(t) \to \boldsymbol{X}_0$ *in the* $C^{1,\gamma}$ *norm as* $t \to 0$*.*

In item (i) above, if we require that $\boldsymbol{u}, p$ *satisfy* (1.9) *in the distributional sense instead of* (1.2)-(1.5)*, we say that* $\boldsymbol{X}(t)$ *is a* classical IB solution*.*

Definition 4.2.6 (Classical BI solution)**.** *Let* $\boldsymbol{X} \in C((0,T]; C^2(\mathbb{S}^1)) \cap C^1((0,T]; C(\mathbb{S}^1))$ *and* $\boldsymbol{X}_0 \in C^{1,\gamma}(\mathbb{S}^1)$*,* $0 < \gamma < 1$ *and* $|\boldsymbol{X}_0|_* > 0$*. The function* $\boldsymbol{X}(t)$ *is a* classical BI solution *with initial data* $\boldsymbol{X}_0$ *if it satisfies* (1.13) *for* $0 < t \le T$ *and* $\boldsymbol{X}(t) \to \boldsymbol{X}_0$ *in* $C^{1,\gamma}$ *as* $t \to 0$*.*

Corollary 1.2.5. That the classical jump and IB solutions are equivalent follows from standard arguments. We refer the reader, for example, to [17, 18] for details.

Suppose we have a classical IB solution. We now show that it is a classical BI solution. Let $\boldsymbol{X}(t)$ be a classical IB solution and let $\boldsymbol{u}(\boldsymbol{x}, t)$ and $p(\boldsymbol{x}, t)$ be the corresponding velocity and pressure fields. Define, as in (1.10) and (1.12),

$$
\begin{aligned}
\widetilde{\boldsymbol{u}}(\boldsymbol{x}, t) &= \int_{\mathbb{S}^1} G(\boldsymbol{x} - \boldsymbol{X}(\theta', t)) \partial_{\theta'}^2 \boldsymbol{X}(\theta', t) d\theta', \\
\widetilde{p}(\boldsymbol{x}, t) &= \int_{\mathbb{S}^1} \boldsymbol{P}_{\mathrm{st}}(\boldsymbol{x} - \boldsymbol{X}(\theta', t)) \cdot \partial_{\theta'}^2 \boldsymbol{X}(\theta', t) d\theta'.
\end{aligned}
$$

That $\widetilde{\boldsymbol{u}}$ and $\widetilde{p}$ satisfy (1.9) in a distributional sense, again, follows from standard arguments. We have thus only to show that $\widetilde{\boldsymbol{u}} = \boldsymbol{u}$. Once we know that this is true, we may use (1.6) to see that $\boldsymbol{X}$ is indeed a classical BI solution. Consider the functions $\boldsymbol{w} = \boldsymbol{u} - \widetilde{\boldsymbol{u}}$ and $q = p - \widetilde{p}$. The functions $\boldsymbol{w}$ and q satisfy the following equations in the sense of distributions.

$$-\Delta \boldsymbol{w} + \nabla q = 0, \ \nabla \cdot \boldsymbol{w} = 0. \tag{4.30}$$

From this, we see that, in the sense of distributions,

$$\Delta q = 0.$$

By Weyl's theorem q is smooth, and thus q is a harmonic function. By (1.8), q is bounded and thus by Liouville's theorem, reduces to a constant. Substituting this back into (4.30), we see that each component of $\boldsymbol{w}$ is also harmonic. Now, note

$$\widetilde{\boldsymbol{u}}(\boldsymbol{x}, t) = -\int_{\mathbb{S}^1} \partial_{\theta'} G(\boldsymbol{x} - \boldsymbol{X}(\theta', t)) \partial_{\theta'} \boldsymbol{X}(\theta', t) d\theta' \tag{4.31}$$

The kernel $\partial'_\theta G(\boldsymbol{x} - \boldsymbol{X}')$ behaves like $|\boldsymbol{x} - \boldsymbol{X}'|^{-1}$ as $\boldsymbol{x} \to \infty$, and thus, $\widetilde{\boldsymbol{u}}(\boldsymbol{x}) \to 0$ as $\boldsymbol{x} \to \infty$. By (1.8), $\boldsymbol{w}$ is thus bounded and tends to 0 as $\boldsymbol{x} \to \infty$. Thus, $\boldsymbol{w} = 0$.

The steps going from (1.13) to (1.18) are easily justified since $\boldsymbol{X}(t) \in C^2(\mathbb{S}^1)$ for each fixed $t > 0$. A classical BI solution is thus a strong solution. We know from Lemma 4.1.8 that a strong solution is a mild solution.

As established in Theorem 1.2.4, the mild solution $\boldsymbol{X}(t)$ satisfies $\boldsymbol{X}, \partial_t \boldsymbol{X} \in C^\infty(\mathbb{S}^1)$ for every fixed $t > 0$. By Lemma 4.1.7 the mild solutions are strong, and we may follow our inference backwards from (1.18) to (1.13) to see that this solution is in fact a classical BI solution. Standard arguments again show that a classical BI solution with smooth $\boldsymbol{X}$ and $\partial_t \boldsymbol{X}$ for fixed t is a classical jump solution (and therefore, also a classical IB solution).

Finally, we establish the conservation of area and the energy identity. Our strong solution satisfies the jump formulation of the Peskin problem, and from this, we see that (1.14) is a direct consequence of (1.3) and (1.6). To obtain the energy identity (1.24), take the inner product of (1.2) with $\boldsymbol{u}$ and integrate by parts. The boundary term at $\boldsymbol{x} \to \infty$ vanish thanks to the fact that $\nabla \boldsymbol{u}$ behaves like $|\boldsymbol{x}|^{-2}$ as $\boldsymbol{x} \to \infty$. This decay follows directly from the boundary integral representation (1.10), which we may rewrite as in (4.31). $\square$

4.3 Equilibria and Stability

4.3.1 Equilibria

The goal of this Section is to calculate the equilibria of the Peskin system and determine the stability of the equilibria. We first calculate the equilibria, or stationary states, of the problem. Consider any solution $\boldsymbol{X} \in C^{1,\alpha}$ that is stationary. Note first that stationary solutions are classical jump solutions, as established in Corollary 1.2.5. Recall by Proposition 1.2.5 that $\boldsymbol{X}$ must satisfy the energy identity (1.24). Since the $\boldsymbol{X}$ is stationary, the energy $\mathcal{E}$ is not changing in time. Thus, the dissipation $\mathcal{D} = 0$ and thus $\nabla \boldsymbol{u} = 0$. Given (1.8), $\boldsymbol{u} = 0$ everywhere. From the Stokes equation (1.2), we thus conclude that:

$$\nabla p = 0 \text{ in } \Omega_{\mathrm{i,e}}.$$

Thus, the pressure p is constant within each domain. Let these pressure values be $p_{\mathrm{i,e}}$ respectively in $\Omega_{\mathrm{i,e}}$. Then, by the stress boundary condition (1.5), we have:

$$-\triangle p \boldsymbol{n} \left|\partial_\theta \boldsymbol{X}\right| = \partial_\theta^2 \boldsymbol{X},\ \triangle p = p_{\mathrm{i}} - p_{\mathrm{e}},\ \boldsymbol{n} = \left|\partial_\theta \boldsymbol{X}\right|^{-1} R_{\pi/2}^{-1} \partial_\theta \boldsymbol{X},$$

where $\boldsymbol{n}$ is the outward normal (pointing from Ω_{i} to Ω_{e}) and $R_{\pi/2}$ is the 2×2 matrix of rotation by $\pi/2$ in the counter-clockwise direction. For definiteness, we have assumed that the parametrization of the curve is the counter-clockwise direction. This immediately yields:

$$\partial_\theta^2 \boldsymbol{X} = \triangle p R_{\pi/2} \partial_\theta \boldsymbol{X}.$$

This is an easily solved differential equation. Noting that $\boldsymbol{X}(2\pi) = \boldsymbol{X}(0)$, we have $\triangle p = 1$ and

$$\boldsymbol{X}(\theta) = A\boldsymbol{e}_{\mathrm{r}} + B\boldsymbol{e}_{\mathrm{t}} + C_1 \boldsymbol{e}_x + C_2 \boldsymbol{e}_y,$$

where A, B, C_1, C_2 are real numbers and $e_{\mathrm{t,t},x,y}$ were defined in (1.26) The only equilibrium configurations, therefore, are circles with equidistant material points. To make sure that the curve does not degenerate to a point, we impose the condition $A^2 + B^2 > 0$. We have thus proved the following:

Proposition 4.3.1. *The only stationary mild solutions of the Peskin problem are circles with equidistant material points. We may parametrize this set* $\widehat{\mathcal{V}}$ *as in* (1.26).

The above proposition is part of Theorem 1.2.6, but we have restated this here for easier reference.

4.3.2 Spectral and Linear Stability

To study the stability of these equilibria, we linearize our equation around the stationary solutions. We study the spectrum of the resulting linear operator $\mathcal{L}$ and the decay properties of $e^{t\mathcal{L}}$, the semigroup operator generated by $\mathcal{L}$. The linearized operator of equation (1.18) at $\boldsymbol{X}$ is given by:

$$\mathcal{L}_{\boldsymbol{X}}\boldsymbol{Y} = \frac{d}{d\epsilon}\left(\Lambda(\boldsymbol{X}+\epsilon\boldsymbol{Y}) + \mathcal{R}(\boldsymbol{X}+\epsilon\boldsymbol{Y})\right)\Big|_{\epsilon=0} = \Lambda\boldsymbol{Y} + \partial_{\boldsymbol{X}}\mathcal{R}(\boldsymbol{X})\boldsymbol{Y}.$$

The derivative $\partial_{\boldsymbol{X}}\mathcal{R}$ was already introduced in Proposition 4.1.5. Let us linearize around the unit circle:

$$\boldsymbol{X}_\star(\theta) = \begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}.$$

We may compute the linearization around $\boldsymbol{X}_\star$ using expressions (1.13) and (1.17). This linearization $\mathcal{L} = \mathcal{L}_{\boldsymbol{X}_\star}$ acting on $\boldsymbol{Y}$ is given by:

$$\begin{aligned}
\mathcal{L}\boldsymbol{Y} &= \Lambda\boldsymbol{Y} + \partial_{\boldsymbol{X}}\mathcal{R}(\boldsymbol{X}_\star)\boldsymbol{Y} \\
&= \int_{\mathbb{S}^1}\left(G(\Delta\boldsymbol{X}_\star)\partial_{\theta'}^2\boldsymbol{Y}' + (\partial_{\boldsymbol{X}}G(\Delta\boldsymbol{X}_\star)\Delta\boldsymbol{Y})\partial_{\theta'}^2\boldsymbol{X}_\star'\right)d\theta', \\
\partial_{\boldsymbol{X}}G(\Delta\boldsymbol{X}_\star)\Delta\boldsymbol{Y} &= -\frac{\Delta\boldsymbol{X}_\star\cdot\Delta\boldsymbol{Y}}{|\Delta\boldsymbol{X}_\star|^2}I + \frac{\Delta\boldsymbol{Y}\otimes\Delta\boldsymbol{X}_\star}{|\Delta\boldsymbol{X}_\star|^2} \\
&\quad + \frac{\Delta\boldsymbol{X}_\star\otimes\Delta\boldsymbol{Y}}{|\Delta\boldsymbol{X}_\star|^2} - 2\frac{\Delta\boldsymbol{X}_\star\cdot\Delta\boldsymbol{Y}}{|\Delta\boldsymbol{X}_\star|^2}\frac{\Delta\boldsymbol{X}_\star\otimes\Delta\boldsymbol{X}_\star}{|\Delta\boldsymbol{X}_\star|^2}.
\end{aligned} \tag{4.32}$$

In fact, given the translation, rotation and dilation symmetries discussed in Section 1.2 (see (1.25) and surrounding discussion) linearization around any circular equilibrium will produce the same linearization. Thus:

$$\mathcal{L}_{\boldsymbol{Z}} = \mathcal{L}_{\boldsymbol{X}_\star} = \mathcal{L} \text{ for any } \boldsymbol{Z} \text{ that is a circular stationary state.} \tag{4.33}$$

We introduce some notation. Define the projection operators:

$$\mathcal{P}_{\mathrm{trl}}\boldsymbol{w} = \frac{1}{2\pi}\left(\langle\boldsymbol{w},\boldsymbol{e}_x\rangle\boldsymbol{e}_x + \langle\boldsymbol{w},\boldsymbol{e}_y\rangle\boldsymbol{e}_y\right),\ \Pi_{\mathrm{trl}}\boldsymbol{w} = \boldsymbol{w} - \mathcal{P}_{\mathrm{trl}}\boldsymbol{w}, \tag{4.34}$$

where the inner product $\langle\cdot,\cdot\rangle$ was introduced in (1.27). The projection $\mathcal{P}_{\mathrm{trl}}$ extracts the translation component of the function (or curve) $\boldsymbol{w}$. After a rather long calculation, we find that (4.32) can be expressed as follows:

$$\begin{aligned}\mathcal{L}\boldsymbol{Y} &= R_\theta \Lambda R_\theta^{-1}\boldsymbol{Y} + \frac{1}{4}\mathcal{P}_{\mathrm{trl}}\boldsymbol{Y} = R_\theta \Lambda R_\theta^{-1}\Pi_{\mathrm{trl}}\boldsymbol{Y},\\ (R_\theta \Lambda R_\theta^{-1}\boldsymbol{w})(\theta) &= \begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}\Lambda & 0\\ 0 & \Lambda\end{pmatrix}\begin{pmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}u(\theta)\\ v(\theta)\end{pmatrix},\end{aligned} \tag{4.35}$$

where

$$\boldsymbol{w} = \begin{pmatrix}u\\ v\end{pmatrix}.$$

Here, Λ is the familiar operator that first appeared in (1.18). The 2×2 rotation matrices R_θ and R_θ^{-1} act simply by matrix vector multiplication. From this expression, the eigenvalues of $\mathcal{L}$ are easily obtained, which are given by:

$$\lambda_k = -\frac{k}{4},\ k = \{0\} \cup \mathbb{N}.$$

For each eigenvalue, the eigenvectors for are given as follows. The eigenspace for $\lambda_0 = 0$ is the span generated by:

$$\lambda_0 = 0: \quad \boldsymbol{e}_{\mathrm{r}},\ \boldsymbol{e}_{\mathrm{t}},\ \boldsymbol{e}_x,\ \boldsymbol{e}_y, \tag{4.36}$$

where the vectors $\boldsymbol{e}_{\mathrm{r,t}}, \boldsymbol{e}_{x,y}$ were defined in (1.26). The eigenspace $\mathcal{V}$ for $\lambda_0 = 0$ coincide with the set of circular equilibria $\widehat{\mathcal{V}}$ (except for the non-degeneracy condition) and reflect the group symmetries of our system. The vector $\boldsymbol{e}_{\mathrm{r,t}}$ correspond to dilation and rotational symmetry respectively and $\boldsymbol{e}_{x,y}$ with translational symmetry. For $\lambda_1 = -1/4$, the two-dimensional eigenspace is spanned by:

$$\lambda_1 = -\frac{1}{4}: \quad \begin{pmatrix}\cos(2\theta)\\ \sin(2\theta)\end{pmatrix},\ \begin{pmatrix}-\sin(2\theta)\\ \cos(2\theta)\end{pmatrix}. \tag{4.37}$$

For $\lambda_k = -k/4,\ k \geq 2$, the four-dimensional eigenspace is spanned by:

$$\lambda_k = -\frac{k}{4},\ k \geq 2: \quad \cos(k\theta)\boldsymbol{e}_{\mathrm{r}},\ \sin(k\theta)\boldsymbol{e}_{\mathrm{r}},\ \cos(k\theta)\boldsymbol{e}_{\mathrm{t}},\ \sin(k\theta)\boldsymbol{e}_{\mathrm{t}}.$$

Of the above four eigenmodes, the two eigenmodes proportional to $\boldsymbol{e}_{\mathrm{r}}$ correspond to radial deformations with a change in shape from the circular configuration, whereas the other two eigenmodes proportional to $\boldsymbol{e}_{\mathrm{t}}$ correspond to tangential deformations without change in circular shape. The eigenvectors corresponding to the above eigenvalues can be easily checked to form a complete orthogonal set in $L^2(\mathbb{S}^1;\mathbb{R}^2)$, and thus the above list exhausts the spectrum of $\mathcal{L}$ as an operator on $L^2(\mathbb{S}^1;\mathbb{R}^2)$. We have only obtained the spectrum of $\mathcal{L}$ as an operator defined in $L^2(\mathbb{S}^1)$, not as a closed operator on $C^{1,\gamma}(\mathbb{S}^1)$.It is not difficult to show that the $\mathcal{L}$ has the same spectrum in $C^{1,\gamma}(\mathbb{S}^1)$. We will, however, not need this information here since the above spectral structure will allow us to explicitly compute the semigroup operator $e^{t\mathcal{L}}$, from which mapping properties of the semigroup operator $e^{t\mathcal{L}}$ in $C^{1,\gamma}(\mathbb{S}^1)$ will be determined directly.

It is interesting that the spectral structure of $\mathcal{L}$ and Λ (acting component-wise on $\mathbb{R}^2$ valued functions) are very similar. The difference between the two can be written in terms of the Hilbert transform:

$$\mathcal{L}\boldsymbol{w} = \Lambda\boldsymbol{w} + \frac{1}{4}\begin{pmatrix} 0 & -\mathcal{H} \\ \mathcal{H} & 0 \end{pmatrix}\boldsymbol{w}. \tag{4.38}$$

The calculation of the above spectrum around a stationary circle is essentially due to [8], where the authors treat the problem in which the Navier-Stokes equation replaces the Stokes equation in our problem. For a Navier-Stokes fluid, a boundary integral reduction is not available and they instead base their calculation on the linearization of the jump and IB formulation of the equations. Our results can be obtained by setting the Reynolds number (or mass density of the fluid) to 0 in their results, except that they seemed to have missed the principal non-zero eigenspace/eigenvector corresponding to λ_1 above.

Except for the eigenvalues corresponding to the four-dimensional group of symmetries, all eigenvalues are negative. In this sense, we may say that the uniform circular configurations are spectrally stable. Our next result shows that the linear evolution semigroup $e^{t\mathcal{L}}$ has decay properties expected from this spectral structure of $\mathcal{L}$. This may be understood as a linear stability result. Recall the projection operators introduced in (1.28). It is easily checked that:

$$\mathcal{L} = R_\theta \Lambda R_\theta^{-1}\Pi = \Pi R_\theta \Lambda R_\theta^{-1}.$$

Let $e^{t\mathcal{L}}$ be the evolution operator generated by $\mathcal{L}$. From the above, we see that

$$e^{t\mathcal{L}}\boldsymbol{w} = R_\theta e^{t\Lambda} R_\theta^{-1}\Pi\boldsymbol{w} + \mathcal{P}\boldsymbol{w}.$$

We have the following exponential decay estimate for $e^{t\mathcal{L}}$.

Proposition 4.3.2. *Let $\boldsymbol{w} \in C^\alpha(\mathbb{S}^1), \alpha \geq 0$ and let $\beta \geq 0$ satisfy $0 \leq \beta - \alpha \leq 1$. Then,*

$$\left\|e^{t\mathcal{L}}\Pi\boldsymbol{w}\right\|_{C^\beta} \leq Ce^{-t/4}\left(\frac{1}{t^{\beta-\alpha}} + 1\right)\|\Pi\boldsymbol{w}\|_{C^\alpha},\ t > 0,$$

where the constant C above depends only on α and β.

Proof. Note first that, clearly $R_\theta, R_\theta^{-1}, \Pi$ are bounded operators from C^α to itself for any $\alpha \geq 0$. Thus, for $t \leq 4\ln 2$, say,

$$\begin{aligned} \left\|e^{t\mathcal{L}}\Pi\boldsymbol{w}\right\|_{C^\beta} &= \left\|R_\theta e^{t\Lambda}R_\theta^{-1}\Pi\boldsymbol{w}\right\|_{C^\beta} \leq C\left\|e^{t\Lambda}R_\theta^{-1}\Pi\boldsymbol{w}\right\|_{C^\beta} \\ &\leq \frac{C}{t^{\beta-\alpha}}\left\|R_\theta^{-1}\Pi\boldsymbol{w}\right\|_{C^\alpha} \leq \frac{C}{t^{\beta-\alpha}}\left\|\Pi\boldsymbol{w}\right\|_{C^\alpha}. \end{aligned} \tag{4.39}$$

where we used Proposition 3.1.7 in the second inequality.

We turn to the estimate for $t \geq 4\ln 2$. Introduce the following projection acting on scalar valued functions on $\mathbb{S}^1$:

$$\widehat{\mathcal{P}}w = \frac{1}{2\pi}\langle w, 1\rangle, \; \widehat{\Pi}w = w - \widehat{\mathcal{P}}w.$$

where $\langle\cdot,\cdot\rangle$ is the standard L^2 inner product. It can then be easily checked that:

$$e^{t\mathcal{L}}\Pi\boldsymbol{w} = R_\theta e^{t\Lambda}\widehat{\Pi}R_\theta^{-1}\Pi\boldsymbol{w}. \tag{4.40}$$

In the above, the operator $e^{t\Lambda}\widehat{\Pi}$ acts component-wise. Let us examine the action of this operator. The projection $\widehat{P}$ is the spectral projection for $e^{t\Lambda}$ for the eigenvalue 0. Performing a calculation similar to (3.9) we have:

$$(e^{t\Lambda}\widehat{\Pi}w)(\theta) = \frac{1}{2\pi}\int_{\mathbb{S}^1} e^{-t/4}\widehat{P}(e^{-t/4}, \theta-\theta')w(\theta')d\theta', \; \widehat{P}(r,\theta) = \frac{2(\cos(\theta)-r)}{1-2r\cos\theta+r^2}.$$

Using the fact that $\exp(-t/4) \leq 1/2$ for $t \geq 4\ln 2$, it is easily seen that:

$$\int_{\mathbb{S}^1}\left|\widehat{P}(e^{-t/4},\theta)\right|d\theta \leq C, \; \int_{\mathbb{S}^1}\left|\partial_\theta\widehat{P}(e^{-t/4},\theta)\right|d\theta \leq C,$$

where the above C are constants independent of t, so long as $t \geq 4\ln 2$. This then immediately shows, in a manner similar to Proposition 3.1.6 that:

$$\left\|e^{t\Lambda}\widehat{\Pi}w\right\|_{C^k} \leq Ce^{-t/4}\left\|w\right\|_{C^k}, \quad \left\|e^{t\Lambda}\widehat{\Pi}w\right\|_{C^{k+1}} \leq Ce^{-t/4}\left\|w\right\|_{C^k}, \quad t \geq 4\ln 2.$$

for $k = \{0\} \cup \mathbb{N}$. We may now use Proposition 3.1.8 as in the proof of Proposition 3.1.7 to obtain:

$$\left\|e^{t\Lambda}\widehat{\Pi}w\right\|_{C^\beta} \leq Ce^{-t/4}\left\|w\right\|_{C^\alpha}, \; 0 \leq \beta-\alpha \leq 1, \; t \geq 4\ln 2.$$

Using this, we may estimate (4.40) as follows.

$$\begin{aligned} \left\|e^{t\mathcal{L}}\Pi\boldsymbol{w}\right\|_{C^\beta} &= \left\|R_\theta e^{t\Lambda}\widehat{\Pi}R_\theta^{-1}\Pi\boldsymbol{w}\right\|_{C^\beta} \\ &\leq C\left\|e^{t\Lambda}\widehat{\Pi}R_\theta^{-1}\Pi\boldsymbol{w}\right\|_{C^\beta} \leq Ce^{-t/4}\left\|R_\theta^{-1}\Pi\boldsymbol{w}\right\|_{C^\alpha} \leq Ce^{-t/4}\left\|\Pi\boldsymbol{w}\right\|_{C^\alpha}. \end{aligned} \tag{4.41}$$

Combining estimate (4.39) valid for $t \leq 4\ln 2$ and the above estimate valid for $t \geq 4\ln 2$, we obtain the desired estimate. $\square$

4.3.3 Nonlinear Stability

The goal of this subsection is to prove Theorem 1.2.6. Note that equation (1.18) can be written as follows:

$$\partial_t \boldsymbol{X} = \mathcal{L}\boldsymbol{X} + \mathcal{N}(\boldsymbol{X}),\ \mathcal{N}(\boldsymbol{X}) = \Lambda \boldsymbol{X} + \mathcal{R}(\boldsymbol{X}) - \mathcal{L}\boldsymbol{X} \tag{4.42}$$

Our strategy is similar to the one we used to establish the existence/uniqueness of mild solutions of the Peskin problem; we turn the above into an integral equation as in (1.19) and use the linear estimate in Proposition 4.3.2 to prove exponential decay to circular equilibria. This is a standard technique used to study stability of equilibria in ODEs and in parabolic problems, and is sometimes referred to as the Lyapunov-Perron method [31]. In order to establish nonlinear stability, we must obtain estimates on the remainder term $\mathcal{N}$.

Recall from Section 1.2 that $\mathcal{V}$ is the kernel of $\mathcal{L}$, or the four-dimensional eigenspace corresponding to the eigenvalue 0 spanned by (4.36). According to Proposition 4.3.1, the set $\widehat{\mathcal{V}} \subset \mathcal{V}$ corresponds to the set of equilibria. Take any point $\boldsymbol{Z} \in \widehat{\mathcal{V}}$. Since $\boldsymbol{Z}$ is a stationary mild solution (and is, consequently, a strong solution)

$$0 = \mathcal{L}\boldsymbol{Z} + \mathcal{N}(\boldsymbol{Z}) = \mathcal{N}(\boldsymbol{Z}),$$

where we used the fact that $\mathcal{L}\boldsymbol{Z} = 0$ since $\boldsymbol{Z} \in \widehat{\mathcal{V}} \subset \mathcal{V}$. Furthermore,

$$\partial_{\boldsymbol{X}} \mathcal{N}[\boldsymbol{Z}]\boldsymbol{Y} = \Lambda \boldsymbol{Y} + \partial_{\boldsymbol{X}} \mathcal{R}[\boldsymbol{Z}]\boldsymbol{Y} - \mathcal{L}\boldsymbol{Y} = 0.$$

This follows from the definition of $\mathcal{L}$ in (4.32) and the fact that the linearization at all stationary circles are the same, see (4.33). For later convenience, let us restate these observations:

$$\mathcal{N}(\boldsymbol{Z}) = 0 \text{ and } \partial_{\boldsymbol{X}} \mathcal{N}[\boldsymbol{Z}]\boldsymbol{Y} = 0 \text{ for } \boldsymbol{Z} \in \widehat{\mathcal{V}},\ \boldsymbol{Y} \in C^{1,\gamma}(\mathbb{S}^1). \tag{4.43}$$

Our next step is to establish the Lipschitz continuity of $\partial_{\boldsymbol{X}} \mathcal{R}$.

Proposition 4.3.3. *Given any $M \geq m > 0$ and convex set $\mathcal{B} \subset O^{M,m} = \{\boldsymbol{Y} \in C^{1,\gamma} : \|\boldsymbol{Y}\|_{C^{1,\gamma}} \leq M$ and $|\boldsymbol{Y}|_* \geq m\}$, with $0 < \gamma < 1$, and $\boldsymbol{V}, \boldsymbol{W}$ and $\boldsymbol{Z} \in \mathcal{B}$, if $\gamma \neq 1/2$ we have:*

$$\|\partial_{\boldsymbol{X}} \mathcal{R}(\boldsymbol{V})\boldsymbol{Z} - \partial_{\boldsymbol{X}} \mathcal{R}(\boldsymbol{W})\boldsymbol{Z}\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} \leq C\frac{M^4}{m^5} \|\boldsymbol{V} - \boldsymbol{W}\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}.$$

If $\gamma = 1/2$, we have for any $\alpha \in (0,1)$

$$\|\partial_{\boldsymbol{X}} \mathcal{R}(\boldsymbol{V})\boldsymbol{Z} - \partial_{\boldsymbol{X}} \mathcal{R}(\boldsymbol{W})\boldsymbol{Z}\|_{C^{0,\alpha}} \leq C\frac{M^4}{m^5} \|\boldsymbol{V} - \boldsymbol{W}\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}.$$

Proof. We prove only the $\gamma \neq 1/2$ case as the $\gamma = 1/2$ case is easily derived by adapting the $\gamma < 1/2$ case. Define:

$$\partial^2_{\boldsymbol{X}} \mathcal{R}[\boldsymbol{Y}](\boldsymbol{V})(\boldsymbol{Z}) := \frac{d}{d\epsilon} \left. \left(\partial_{\boldsymbol{X}} \mathcal{R}(\boldsymbol{Y} + \epsilon \boldsymbol{V})\boldsymbol{Z}\right)\right|_{\epsilon=0}.$$

We show that, if $\boldsymbol{Y} \in C^{1,\gamma}$ and $|\boldsymbol{Y}|_* > 0$, the following estimate holds:

$$\left\|\partial^2_{\boldsymbol{X}} \mathcal{R}[\boldsymbol{Y}](\boldsymbol{V})(\boldsymbol{Z})\right\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} \leq C \frac{\|\boldsymbol{Y}\|^4_{C^{1,\gamma}}}{|\boldsymbol{Y}|^5_*} \|\boldsymbol{V}\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}. \tag{4.44}$$

Once we have this estimate, the desired bound is immediate, for:

$$\begin{aligned}
&\|\partial_{\boldsymbol{X}} \mathcal{R}(\boldsymbol{V})\boldsymbol{Z} - \partial_{\boldsymbol{X}} \mathcal{R}(\boldsymbol{W})\boldsymbol{Z}\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} \\
&\quad = \left\| \int_0^1 \frac{d}{ds} \partial_{\boldsymbol{X}} \mathcal{R}((1-s)\boldsymbol{V} + s\boldsymbol{W})\boldsymbol{Z} ds \right\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} \\
&\quad \leq \int_0^1 \left\|\partial^2_{\boldsymbol{X}} \mathcal{R}[(1-s)\boldsymbol{V} + s\boldsymbol{W}](\boldsymbol{V} - \boldsymbol{W})(\boldsymbol{Z})\right\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} ds \\
&\quad \leq C \frac{M^4}{m^5} \|\boldsymbol{V} - \boldsymbol{W}\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}},
\end{aligned}$$

where we used the assumption that $\mathcal{B}$ is convex in the last inequality.

Recall from (4.1) that

$$\mathcal{R}(\boldsymbol{X}) = \mathcal{R}_C(\boldsymbol{X}) + \mathcal{R}_T(\boldsymbol{X}). \tag{4.45}$$

We thus have:

$$\partial_{\boldsymbol{X}} \mathcal{R}(\boldsymbol{X})\boldsymbol{Z} = \partial_{\boldsymbol{X}} \mathcal{R}_C(\boldsymbol{X})\boldsymbol{Z} + \partial_{\boldsymbol{X}} \mathcal{R}_T(\boldsymbol{X})\boldsymbol{Z}.$$

and

$$\partial^2_{\boldsymbol{X}} \mathcal{R}(\boldsymbol{X})\boldsymbol{Z} = \partial^2_{\boldsymbol{X}} \mathcal{R}_C(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z} + \partial^2_{\boldsymbol{X}} \mathcal{R}_T(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z}.$$

We first focus on $\partial^2_{\boldsymbol{X}} \mathcal{R}_C(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z}$. Note that lemma 2.2.8 implies that

$$
\begin{aligned}
\partial_{\boldsymbol{X}} \mathcal{R}_C(\boldsymbol{Y})\boldsymbol{Z} &= \frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'}\left(\partial_{\boldsymbol{X}}\left(\log|\Delta \boldsymbol{Y}|\right)\boldsymbol{Z}\right)\partial_{\theta'}\boldsymbol{Y}\, d\theta' \\
&\quad + \frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'}\left(\log|\Delta \boldsymbol{Y}|\right)\partial_{\theta'}\boldsymbol{Z}' d\theta' \\
&= \frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'}\left(\frac{\Delta \boldsymbol{Y}\cdot\Delta \boldsymbol{Z}}{|\Delta \boldsymbol{Y}|^2}\right)\partial_{\theta'}\boldsymbol{Y}\, d\theta' \\
&\quad + \frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'}\left(\log|\Delta \boldsymbol{Y}|\right)\partial_{\theta'}\boldsymbol{Z}' d\theta'
\end{aligned}
$$

so that

$$
\begin{aligned}
\partial^2_{\boldsymbol{X}} \mathcal{R}_C(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z} &= \frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'}\left(\partial_{\boldsymbol{X}}\left(\frac{\Delta \boldsymbol{Y}\cdot\Delta \boldsymbol{Z}}{|\Delta \boldsymbol{Y}|^2}\right)\boldsymbol{W}\right)\partial_{\theta'}\boldsymbol{Y}\, d\theta' \\
&\quad + \frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'}\left(\frac{\Delta \boldsymbol{Y}\cdot\Delta \boldsymbol{Z}}{|\Delta \boldsymbol{Y}|^2}\right)\partial_{\theta'}\boldsymbol{W}\, d\theta' \\
&\quad + \frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'}\left(\frac{\Delta \boldsymbol{Y}\cdot\Delta \boldsymbol{W}}{|\Delta \boldsymbol{Y}|^2}\right)\partial_{\theta'}\boldsymbol{Z}' d\theta' \\
&=: \mathcal{R}^1_C(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z} + \mathcal{R}^2_C(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z} + \mathcal{R}^3_C(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z}
\end{aligned}
$$

Note that lemma 2.2.6 implies that the kernel of the first term is a sum of functions f_k in class $\mathcal{S}^H_{0,1,\gamma}$. Thus $\partial^2_{\boldsymbol{X}} \mathcal{R}_C(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z} \in C^{\lfloor 2\gamma\rfloor, 2\gamma-\lfloor 2\gamma\rfloor}$. This lemma further implies that each f_k has $N = 2$. Thus, proposition 2.3.1 yields

$$
\begin{aligned}
\left\|\partial^2_{\boldsymbol{X}} \mathcal{R}^1_C(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z}\right\|_{C^{\lfloor 2\gamma\rfloor, 2\gamma-\lfloor 2\gamma\rfloor}} &\leq C\frac{\|\boldsymbol{Y}\|^4_{C^{1,\gamma}}}{|\boldsymbol{Y}|^5_*}\|\boldsymbol{W}\|_{C^{1,\gamma}}\|\boldsymbol{Z}\|_{C^{1,\gamma}} \\
&\leq C\frac{M^4}{m^5}\|\boldsymbol{W}\|_{C^{1,\gamma}}\|\boldsymbol{Z}\|_{C^{1,\gamma}}.
\end{aligned}
$$

We may apply proposition 2.3.1 to the other two terms and obtain the same bound as $M > m$.

We can repeat this process for term $\mathcal{R}_T$. We have,

$$
\begin{aligned}
\partial_{\boldsymbol{X}} \mathcal{R}_T(\boldsymbol{Y})\boldsymbol{Z} &= -\frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'}\left(\partial_{\boldsymbol{X}}\left(\frac{\Delta \boldsymbol{Y}\otimes \boldsymbol{Y}}{|\Delta \boldsymbol{Y}|^2}\right)\boldsymbol{Z}\right)\partial_{\theta'}\boldsymbol{Y}\, d\theta' \\
&\quad - \frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'}\left(\frac{\Delta \boldsymbol{Y}\otimes \boldsymbol{Y}}{|\Delta \boldsymbol{Y}|^2}\right)\partial_{\theta'}\boldsymbol{Z}\, d\theta'.
\end{aligned}
$$

Lemma 2.2.6 implies that the kernel of the first term is a sum of functions f_k in $\mathcal{S}^H_{0,1,\gamma}$ with $N = 1$. We may linearize again and find

$$\begin{aligned}
\partial^2_{\boldsymbol{X}} \mathcal{R}_T(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z} = &-\frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'} \left(\partial^2_{\boldsymbol{X}} \left(\frac{\Delta \boldsymbol{Y} \otimes \boldsymbol{Y}}{|\Delta \boldsymbol{Y}|^2}\right)[\boldsymbol{W}]\boldsymbol{Z}\right) \partial_{\theta'} \boldsymbol{Y} d\theta' \\
&- \frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'} \left(\partial_{\boldsymbol{X}} \left(\frac{\Delta \boldsymbol{Y} \otimes \boldsymbol{Y}}{|\Delta \boldsymbol{Y}|^2}\right)\boldsymbol{Z}\right) \partial_{\theta'} \boldsymbol{W} d\theta' \\
&- \frac{1}{4\pi}\int_{\mathbb{S}^1} \partial_{\theta'} \left(\partial_{\boldsymbol{X}} \left(\frac{\Delta \boldsymbol{Y} \otimes \boldsymbol{Y}}{|\Delta \boldsymbol{Y}|^2}\right)\right) \partial_{\theta'} \boldsymbol{Z} d\theta' \\
&=: \mathcal{R}^1_T(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z} + \mathcal{R}^2_T(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z} + \mathcal{R}^3_T(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z}
\end{aligned}$$

Another application of lemma 2.2.6 implies that the kernel of the first term is again a sum of functions in $\mathcal{S}^H_{0,1,\gamma}$ but this time, with $N = 2$. Thus, proposition 2.3.1 implies

$$\begin{aligned}
\left\|\partial^2_{\boldsymbol{X}} \mathcal{R}^1_T(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z}\right\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} &\leq C\frac{\|\boldsymbol{Y}\|^4_{C^{1,\gamma}}}{|\boldsymbol{Y}|^5_*} \|\boldsymbol{W}\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}} \\
&\leq C\frac{M^4}{m^5} \|\boldsymbol{W}\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}}.
\end{aligned}$$

We can find the same bounds for $\mathcal{R}^2_C(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z}$ and $\mathcal{R}^3_C(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z}$ as well. Finally, combining these bounds,

$$\left\|\partial^2_{\boldsymbol{X}} \mathcal{R}(\boldsymbol{Y})[\boldsymbol{W}]\boldsymbol{Z}\right\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} \leq C\frac{M^4}{m^5} \|\boldsymbol{W}\|_{C^{1,\gamma}} \|\boldsymbol{Z}\|_{C^{1,\gamma}},$$

as desired.

□

Using the above Proposition together with observation (4.43), we obtain the following estimate on $\mathcal{N}$.

Lemma 4.3.4. *Suppose $\mathcal{B}$ is a convex set contained in $O^{M,m}$ and $\boldsymbol{X}_1, \boldsymbol{X}_2, \mathcal{P}\boldsymbol{X}_1, \mathcal{P}\boldsymbol{X}_2 \in \mathcal{B}$, where $\mathcal{P}$ is the projection given in* (1.28)*. Then, for $\gamma \in (0,1), \gamma \neq 1/2$,*

$$\|\mathcal{N}(\boldsymbol{X}_1) - \mathcal{N}(\boldsymbol{X}_2)\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} \leq C\frac{M^4}{m^5} \left(\|\Pi \boldsymbol{X}_1\|_{C^{1,\gamma}} + \|\Pi \boldsymbol{X}_2\|_{C^{1,\gamma}}\right) \|\boldsymbol{X}_1 - \boldsymbol{X}_2\|_{C^{1,\gamma}},$$

where Π *is defined in* (1.28). *If* $\gamma = 1/2$*,*

$$\left\|\mathcal{N}(\boldsymbol{X}_1) - \mathcal{N}(\boldsymbol{X}_2)\right\|_{C^{0,\alpha}} \leq C\frac{M^4}{m^5}\left(\left\|\Pi\boldsymbol{X}_1\right\|_{C^{1,\gamma}} + \left\|\Pi\boldsymbol{X}_2\right\|_{C^{1,\gamma}}\right)\left\|\boldsymbol{X}_1 - \boldsymbol{X}_2\right\|_{C^{1,\gamma}},$$

for any $\alpha \in (0,1)$.

Proof. Let $\boldsymbol{X}_1, \boldsymbol{X}_2$ be as above. Then,

$$\begin{aligned}\mathcal{N}(\boldsymbol{X}_1) - \mathcal{N}(\boldsymbol{X}_2) &= \int_0^1 \frac{d}{ds}\mathcal{N}(s\boldsymbol{X}_1 + (1-s)\boldsymbol{X}_2)ds &(4.46)\\ &= \int_0^1 \partial_{\boldsymbol{X}}\mathcal{N}[s\boldsymbol{X}_1 + (1-s)\boldsymbol{X}_2]\boldsymbol{Y}ds. &(4.47)\end{aligned}$$

where $\boldsymbol{Y} = \boldsymbol{X}_1 - \boldsymbol{X}_2$. Note that $\mathcal{P}\boldsymbol{X}_1, \mathcal{P}\boldsymbol{X}_2$ and hence $s\mathcal{P}\boldsymbol{X}_1 + (1-s)\mathcal{P}\boldsymbol{X}_2$ are in $\widehat{\mathcal{V}}$ (are circular equilibria). Given (4.43), we have:

$$\partial_{\boldsymbol{X}}\mathcal{N}[\mathcal{P}(s\boldsymbol{X}_1 + (1-s)\boldsymbol{X}_2)]\boldsymbol{Y} = 0.$$

Note also that, for any $\boldsymbol{V} \in C^{1,\gamma}$, $|\boldsymbol{V}|_* > 0$ we have:

$$\partial_{\boldsymbol{X}}\mathcal{N}[\boldsymbol{V}]\boldsymbol{Y} = \partial_{\boldsymbol{X}}\mathcal{R}[\boldsymbol{V}]\boldsymbol{Y} - (\mathcal{L}\boldsymbol{Y} - \Lambda\boldsymbol{Y}).$$

In fact, the difference between $\partial_{\boldsymbol{X}}\mathcal{N}$ and $\partial_{\boldsymbol{X}}\mathcal{R}$ can be written in terms of the Hilbert transform as we saw in (4.38). We may thus estimate the integrand in (4.46) as follows:

$$\begin{aligned}&\left\|\partial_{\boldsymbol{X}}\mathcal{N}[s\boldsymbol{X}_1 + (1-s)\boldsymbol{X}_2]\boldsymbol{Y}\right\|_{C^{\lfloor 2\gamma\rfloor, 2\gamma-\lfloor 2\gamma\rfloor}}\\ =&\left\|\partial_{\boldsymbol{X}}\mathcal{N}[s\boldsymbol{X}_1 + (1-s)\boldsymbol{X}_2]\boldsymbol{Y} - \partial_{\boldsymbol{X}}\mathcal{N}[\mathcal{P}(s\boldsymbol{X}_1 + (1-s)\boldsymbol{X}_2)]\boldsymbol{Y}\right\|_{C^{\lfloor 2\gamma\rfloor, 2\gamma-\lfloor 2\gamma\rfloor}}\\ =&\left\|\partial_{\boldsymbol{X}}\mathcal{R}[s\boldsymbol{X}_1 + (1-s)\boldsymbol{X}_2]\boldsymbol{Y} - \partial_{\boldsymbol{X}}\mathcal{R}[\mathcal{P}(s\boldsymbol{X}_1 + (1-s)\boldsymbol{X}_2)]\boldsymbol{Y}\right\|_{C^{\lfloor 2\gamma\rfloor, 2\gamma-\lfloor 2\gamma\rfloor}}\\ \leq& C\frac{M^4}{m^5}\left\|s\Pi\boldsymbol{X}_1 + (1-s)\Pi\boldsymbol{X}_2\right\|_{C^{1,\gamma}}\left\|\boldsymbol{Y}\right\|_{C^{1,\gamma}}\\ \leq& C\frac{M^4}{m^5}(s\left\|\Pi\boldsymbol{X}_1\right\|_{C^{1,\gamma}} + (1-s)\left\|\Pi\boldsymbol{X}_2\right\|_{C^{1,\gamma}})\left\|\boldsymbol{Y}\right\|_{C^{1,\gamma}}\end{aligned} \tag{4.48}$$

In the first inequality, we used Proposition 4.3.3 and the definition of the projection Π given in (1.28). The desired estimate is now immediate by taking the $C^{\lfloor 2\gamma\rfloor, 2\gamma-\lfloor 2\gamma\rfloor}$ norm on both sides of (4.46) and using (4.48). The second statement follows by applying the second statement of Proposition 4.3.3. $\square$

Recall that $\mathcal{V}$ was the four-dimensional kernel of $\mathcal{L}$ spanned by (4.36). Let $\mathcal{W}$ be the orthogonal complement of $\mathcal{V}$ in $C^{1,\gamma}(\mathbb{S}^1)$:

$$\mathcal{W} = \{\boldsymbol{w} \in C^{1,\gamma}(\mathbb{S}^1) | \langle \boldsymbol{w}, \boldsymbol{v} \rangle = 0 \text{ for all } \boldsymbol{v} \in \mathcal{V}\}.$$

An equivalent definition is that $\mathcal{W}$ are the elements of $C^{1,\gamma}(\mathbb{S}^1)$ annihilated by the projection $\mathcal{P}$. The subspaces $\mathcal{W}$ and $\mathcal{V}$ are both closed in $C^{1,\gamma}(\mathbb{S}^1)$ and are thus Banach spaces with respect to the $C^{1,\gamma}$ norm.

To motivate the calculation to follow, we perform the following formal calculation. Consider (4.42), apply the projections $\mathcal{P}$ and Π and set $\boldsymbol{Y} = \mathcal{P}\boldsymbol{X}$ and $\boldsymbol{Z} = \Pi\boldsymbol{X}$. We see that $\boldsymbol{Y}$ and $\boldsymbol{Z}$ should satisfy:

$$\begin{aligned}
\partial_t \boldsymbol{Y} &= \mathcal{L}Y + \Pi\mathcal{N}(\boldsymbol{Y} + \boldsymbol{Z}), \\
\partial_t \boldsymbol{Z} &= \mathcal{P}\mathcal{N}(\boldsymbol{Y} + \boldsymbol{Z}).
\end{aligned}$$

The equation for $\boldsymbol{Y}$ is a evolution equation in $\mathcal{W}$ whereas the equation for $\boldsymbol{Z}$ lives in $\mathcal{V}$. Given the decay estimate of Proposition 4.3.2 for $e^{t\mathcal{L}}$ acting on $\mathcal{W}$, we expect $\boldsymbol{Y}$ to decay exponentially.

Define the following function spaces:

$$\begin{aligned}
\mathcal{W}_\sigma &= \{\boldsymbol{Y}(t) \in C([0,\infty); \mathcal{W}) | \, \|\boldsymbol{Y}\|_\sigma < \infty\}, \ \|\boldsymbol{Y}\|_\sigma = \sup_{t\geq 0} e^{t\sigma} \|\boldsymbol{Y}(t)\|_{C^{1,\gamma}}, \ \sigma > 0, \\
\mathcal{V}_0 &= \{\boldsymbol{Z}(t) \in C([0,\infty); \mathcal{V}) \, \|\boldsymbol{Z}\|_0 < \infty\}, \ \|\boldsymbol{Z}\|_0 = \sup_{t\geq 0} \|\boldsymbol{Z}(t)\|_{C^{1,\gamma}}.
\end{aligned}$$

Note that $\mathcal{V}$ is finite dimensional, and thus, all norms are equivalent on $\mathcal{V}$. In inequality (1.30) in the statement of Theorem 1.2.6, we used the norm, denoted by $\|\cdot\|_{\mathcal{V}}$, induced by the coordinate vectors $\boldsymbol{e}_{\mathrm{r,t},x,y}$. This is also the norm we use to computationally check our decay result in Section 4.3.4. In the estimates to follow, however, it would be more convenient for us to use the $C^{1,\gamma}$ norm.

For a pair of functions $(\boldsymbol{Y}, \boldsymbol{Z}) \in \mathcal{W}_\sigma \times \mathcal{V}_0$, define the norm to be:

$$\|(\boldsymbol{Y}, \boldsymbol{Z})\|_{\mathcal{W}_\sigma \times \mathcal{V}_0} = \|\boldsymbol{Y}\|_\sigma + \|\boldsymbol{Z}\|_0.$$

Let $\boldsymbol{Z}_\star \in \widehat{\mathcal{V}}$ be a uniformly parametrized circle of radius 1 centered at the origin. Define the set of functions $\mathcal{B}_{M_{\boldsymbol{Y}}, M_{\boldsymbol{Z}}}$ as the set of all $(\boldsymbol{Y}, \boldsymbol{Z}) \in \mathcal{W}_\sigma \times \mathcal{V}_0$ satisfying

$$\|\boldsymbol{Y}\|_\sigma \leq M_{\boldsymbol{Y}} \text{ and } \|\boldsymbol{Z} - \boldsymbol{Z}_\star\|_0 \leq M_{\boldsymbol{Z}}$$

where $M_{\boldsymbol{Y}}$, $M_{\boldsymbol{Z}}$ are some positive constants. In a slight abuse of notation, we let $\boldsymbol{Z}_\star$ above denote a function of t that is constant in time with value $\boldsymbol{Z}_\star$.

Proposition 4.3.5. *Let $0 < \sigma < 1/4$ and $\boldsymbol{Y}_0 \in \mathcal{W}$. Then there exist positive constants $\rho_0, M_{\boldsymbol{Y}}$ and $M_{\boldsymbol{Z}}$ with the following properties. If $\|\boldsymbol{Y}_0\|_{C^{1,\gamma}} \leq \rho_0$, the map*

$$\mathcal{S}\begin{pmatrix}\boldsymbol{Y}\\ \boldsymbol{Z}\end{pmatrix} = \begin{pmatrix} e^{\mathcal{L}t}\boldsymbol{Y}_0 + \int_0^t e^{\mathcal{L}(t-s)}\Pi\mathcal{N}(\boldsymbol{Y}(s)+\boldsymbol{Z}(s))ds \\ \boldsymbol{Z}_\star + \int_0^t \mathcal{P}\mathcal{N}(\boldsymbol{Y}(s)+\boldsymbol{Z}(s))ds\end{pmatrix}$$

maps $\mathcal{B}_{M_{\boldsymbol{Y}},M_{\boldsymbol{Z}}}$ to itself and is a contraction. The resulting fixed point satisfies:

$$\|\boldsymbol{Y}\|_\sigma \leq C\|\boldsymbol{Y}_0\|_{C^{1,\gamma}}, \quad \|\boldsymbol{Z}-\boldsymbol{Z}_\star\|_0 \leq C\|\boldsymbol{Y}_0\|_{C^{1,\gamma}} \tag{4.49}$$

where C is a constant that only depends on σ and γ.

Proof. First, we make $M_{\boldsymbol{Y}}$ and $M_{\boldsymbol{Z}}$ small enough so that perturbations away from $\boldsymbol{Z}_\star$ remains non-degenerate and non-self-intersecting. Consider the curve $\boldsymbol{Z}_\star + \boldsymbol{W}$ where $\boldsymbol{W} \in C^{1,\gamma}(\mathbb{S}^1)$. We have:

$$\begin{aligned}\left|\boldsymbol{Z}_\star(\theta)+\boldsymbol{W}(\theta)-(\boldsymbol{Z}_\star(\theta')+\boldsymbol{W}(\theta'))\right| &\geq \left|\boldsymbol{Z}_\star(\theta)-\boldsymbol{Z}_\star(\theta')\right| - \left|\boldsymbol{W}(\theta)-\boldsymbol{W}(\theta')\right| \\ &\geq \left(|\boldsymbol{Z}_\star|_* - [\boldsymbol{W}]_{C^1}\right)\left|\theta-\theta'\right|.\end{aligned}$$

Dividing through by $|\theta-\theta'|$ and taking the infimum on the left hand side, we have:

$$|\boldsymbol{Z}_\star+\boldsymbol{W}|_* \geq |\boldsymbol{Z}_\star|_* - [\boldsymbol{W}]_{C^1} \geq \frac{2}{\pi} - \|\boldsymbol{W}\|_{C^{1,\gamma}},$$

where we used $|\boldsymbol{Z}_\star|_* = 2/\pi$. If $\|\boldsymbol{W}\|_{C^{1,\gamma}} \leq 1/\pi$, say, we are assured that $|\boldsymbol{Z}_\star+\boldsymbol{W}|_* \geq 1/\pi$. We thus choose $M_{\boldsymbol{Y}}$ and $M_{\boldsymbol{Z}}$ small enough so that

$$\|\boldsymbol{Y}+\boldsymbol{Z}-\boldsymbol{Z}_\star\|_{C^{1,\gamma}} \leq \|\boldsymbol{Y}\|_{C^{1,\gamma}} + \|\boldsymbol{Z}-\boldsymbol{Z}_\star\|_{C^{1,\gamma}} \leq M_{\boldsymbol{Y}} + M_{\boldsymbol{Z}} \leq \frac{1}{\pi}.$$

So long as $M_{\boldsymbol{Y}}$ and $M_{\boldsymbol{Z}}$ are chosen in this way, for $(\boldsymbol{Y}(t),\boldsymbol{Z}(t)) \in \mathcal{B}_{M_{\boldsymbol{Y}},M_{\boldsymbol{Z}}}$ we have:

$$|\boldsymbol{Y}(t)+\boldsymbol{Z}(t)|_* \geq \frac{1}{\pi} = m, \quad \|\boldsymbol{Y}(t)+\boldsymbol{Z}(t)\|_{C^{1,\gamma}} \leq \|\boldsymbol{Z}_\star\|_{C^{1,\gamma}} + \frac{1}{\pi} = M. \tag{4.50}$$

Thus, for any $(\boldsymbol{Y},\boldsymbol{Z}) \in \mathcal{B}_{M_{\boldsymbol{Y}},M_{\boldsymbol{Z}}}$, $\boldsymbol{X}(t) = \boldsymbol{Y}(t)+\boldsymbol{Z}(t) \in O^{M,m}$ with the constants M and m as above.

Let $\boldsymbol{X}_1 = \boldsymbol{Y}_1 + \boldsymbol{Z}_1$ and $\boldsymbol{X}_2 = \boldsymbol{Y}_2 + \boldsymbol{Z}_2$ with $\boldsymbol{Y}_i, \boldsymbol{Z}_i \in \mathcal{B}_{M_{\boldsymbol{Y}},M_{\boldsymbol{Z}}}$ for $i = 1, 2$. Then,

$$\mathcal{S}\begin{pmatrix}\boldsymbol{Y}_1\\ \boldsymbol{Z}_1\end{pmatrix} - \mathcal{S}\begin{pmatrix}\boldsymbol{Y}_2\\ \boldsymbol{Z}_2\end{pmatrix} = \begin{pmatrix}\int_0^t e^{\mathcal{L}(t-s)}\Pi\left(\mathcal{N}(\boldsymbol{X}_1(s)) - \mathcal{N}(\boldsymbol{X}_2(s))\right) ds\\ \int_0^t \mathcal{P}\left(\mathcal{N}(\boldsymbol{X}_1(s)) - \mathcal{N}(\boldsymbol{X}_2(s))\right) ds\end{pmatrix} =: \begin{pmatrix}\boldsymbol{W}_1 - \boldsymbol{W}_2\\ \boldsymbol{V}_1 - \boldsymbol{V}_2\end{pmatrix}.$$

We will show that $\mathcal{S}$ is a contraction in both components. For the first component,

$$\begin{aligned}\|\boldsymbol{W}_1 - \boldsymbol{W}_2\|_\sigma &\leq \sup_{t\geq 0} e^{\sigma t}\int_0^t \left\|e^{\mathcal{L}(t-s)}\Pi\left(\mathcal{N}(\boldsymbol{X}_1(s)) - \mathcal{N}(\boldsymbol{X}_2(s))\right)\right\|_{C^{1,\gamma}} ds\\ &\leq \sup_{t\geq 0} Ce^{\sigma t}\int_0^t e^{-(t-s)/4}\left((t-s)^{-\gamma}+1\right)\|\mathcal{N}(\boldsymbol{X}_1(s)) - \mathcal{N}(\boldsymbol{X}_2(s))\|_{C^{\lfloor 2\gamma\rfloor, 2\gamma-\lfloor 2\gamma\rfloor}} ds,\end{aligned}$$

where we used Proposition 4.3.2 in the second inequality. Using Lemma 4.3.4 and (4.50), we find

$$\|\mathcal{N}(\boldsymbol{X}_1) - \mathcal{N}(\boldsymbol{X}_2)\|_{C^{\lfloor 2\gamma\rfloor, 2\gamma-\lfloor 2\gamma\rfloor}} \leq C\left(\|\boldsymbol{Y}_1\|_{C^{1,\gamma}} + \|\boldsymbol{Y}_2\|_{C^{1,\gamma}}\right)\|\boldsymbol{X}_1 - \boldsymbol{X}_2\|_{C^{1,\gamma}}, \tag{4.51}$$

But for $i = 1, 2$,

$$\|\boldsymbol{Y}_i(s)\|_{C^{1,\gamma}} \leq e^{-\sigma s}\|\boldsymbol{Y}_i\|_{C^{1,\gamma,\sigma}} \leq e^{-\sigma s}M_{\boldsymbol{Y}}.$$

Thus,

$$\begin{aligned}\|\boldsymbol{W}_1 - \boldsymbol{W}_2\|_\sigma &\leq \sup_{t\geq 0} CM_{\boldsymbol{Y}}e^{\sigma t}\int_0^t e^{-(t-s)/4}e^{-\sigma s}\left((t-s)^{-\gamma}+1\right)\|(\boldsymbol{X}_1 - \boldsymbol{X}_2)(s)\|_{C^{1,\gamma}} ds\\ &\leq \sup_{t\geq 0} CM_{\boldsymbol{Y}}\int_0^t e^{-(t-s)(1/4-\sigma)}\left((t-s)^{-\gamma}+1\right) ds\left(\|\boldsymbol{Y}_1 - \boldsymbol{Y}_2\|_\sigma\right)\\ &+ \sup_{t\geq 0} CM_{\boldsymbol{Y}}\int_0^t e^{-(t-s)(1/4-\sigma)}\left((t-s)^{-\gamma}+1\right) ds\left(\|\boldsymbol{Z}_1 - \boldsymbol{Z}_2\|_0\right).\end{aligned}$$

Using $\sigma < 1/4$, we may bound the integral above as follows:

$$\begin{aligned}\int_0^t e^{-(t-s)(1/4-\sigma)}\left((t-s)^{-\gamma}+1\right) ds &= \int_0^t e^{-u(1/4-\sigma)}\left(u^{-\gamma}+1\right) du\\ \leq \int_0^1 (u^{-\gamma}+1)du + \int_1^\infty 2e^{-u(1/4-\sigma)}du &= \frac{1}{1-\gamma} + 1 + \frac{8}{1-4\sigma}.\end{aligned}$$

We thus have:

$$\|\boldsymbol{W}_1 - \boldsymbol{W}_2\|_\sigma \leq CM_{\boldsymbol{Y}}\left(\|\boldsymbol{Y}_1 - \boldsymbol{Y}_2\|_\sigma + \|\boldsymbol{Z}_1 - \boldsymbol{Z}_2\|_0\right).$$

We may shrink $M_{\boldsymbol{Y}}$ as much as we wish so that

$$\|\boldsymbol{W}_1 - \boldsymbol{W}_2\|_\sigma \leq \frac{1}{2}\left(\|\boldsymbol{Y}_1 - \boldsymbol{Y}_2\|_\sigma + \|\boldsymbol{Z}_1 - \boldsymbol{Z}_2\|_0\right). \tag{4.52}$$

We now show that $\mathcal{S}$ is a contraction in the second component as well. We compute

$$\begin{aligned}\|\boldsymbol{V}_1 - \boldsymbol{V}_2\|_0 &\leq \sup_{t\geq 0}\int_0^t \|\mathcal{P}\left(\mathcal{N}(\boldsymbol{X}_1(s)) - \mathcal{N}(\boldsymbol{X}_2(s))\right)\|_{C^{1,\gamma}}\, ds \\ &\leq \sup_{t\geq 0} C\int_0^t \|\mathcal{N}(\boldsymbol{X}_1(s)) - \mathcal{N}(\boldsymbol{X}_2(s))\|_{C^{\lfloor 2\gamma\rfloor, 2\gamma-\lfloor 2\gamma\rfloor}}\, ds,\end{aligned}$$

where we have used the fact that $\mathcal{P}$ is a bounded operator mapping to $\mathcal{V}$ and that all norms are equivalent on a finite dimensional space. Then, using Lemma 4.3.4 gives

$$\begin{aligned}&\|\boldsymbol{V}_1 - \boldsymbol{V}_2\|_0 \\ &\leq \sup_{t\geq 0} C\int_0^t \left(\|\boldsymbol{Y}_1(s)\|_{C^{1,\gamma}} + \|\boldsymbol{Y}_2(s)\|_{C^{1,\gamma}}\right)\left(\|(\boldsymbol{Y}_1 - \boldsymbol{Y}_2)(s)\|_{C^{1,\gamma}} + \|(\boldsymbol{Z}_1 - \boldsymbol{Z}_2)(s)\|_{C^{1,\gamma}}\right) ds \\ &\leq \sup_{t\geq 0} CM_{\boldsymbol{Y}}\int_0^t e^{-\sigma s}\left(\|(\boldsymbol{Y}_1 - \boldsymbol{Y}_2)(s)\|_{C^{1,\gamma}} + \|(\boldsymbol{Z}_1 - \boldsymbol{Z}_2)(s)\|_{C^{1,\gamma}}\right) ds \\ &\leq \sup_{t\geq 0} CM_{\boldsymbol{Y}}\int_0^t \left(e^{-2s\sigma}\|\boldsymbol{Y}_1 - \boldsymbol{Y}_2\|_\sigma + e^{-s\sigma}\|\boldsymbol{Z}_1 - \boldsymbol{Z}_2\|_0\right) ds \\ &\leq CM_{\boldsymbol{Y}}\left(\|\boldsymbol{Y}_1 - \boldsymbol{Y}_2\|_\sigma + \|\boldsymbol{Z}_1 - \boldsymbol{Z}_2\|_0\right).\end{aligned}$$

Shrinking $M_{\boldsymbol{Y}}$ again as needed, we conclude

$$\|\boldsymbol{V}_1 - \boldsymbol{V}_2\|_0 \leq \frac{1}{2}\left(\|\boldsymbol{Y}_1 - \boldsymbol{Y}_2\|_\sigma + \|\boldsymbol{Z}_1 - \boldsymbol{Z}_2\|_0\right). \tag{4.53}$$

We now choose $\|\boldsymbol{Y}_0\|_{C^{1,\gamma}}$ small enough so that $\mathcal{S}$ maps $\mathcal{B}_{M_{\boldsymbol{Y}},M_{\boldsymbol{Z}}}$ to itself.

$$\begin{pmatrix}\boldsymbol{W}\\ \boldsymbol{V}\end{pmatrix} = \mathcal{S}\begin{pmatrix}\boldsymbol{Y}\\ \boldsymbol{Z}\end{pmatrix} = \left(\mathcal{S}\begin{pmatrix}\boldsymbol{Y}\\ \boldsymbol{Z}\end{pmatrix} - \mathcal{S}\begin{pmatrix}0\\ \boldsymbol{Z}\end{pmatrix}\right) + \mathcal{S}\begin{pmatrix}0\\ \boldsymbol{Z}\end{pmatrix}.$$

Note that, since $\mathcal{N}(\boldsymbol{Z}) = 0$ (see (4.43)), we have:

$$\mathcal{S}\begin{pmatrix}0\\ \boldsymbol{Z}\end{pmatrix} = \begin{pmatrix}e^{t\mathcal{L}}\boldsymbol{Y}_0\\ \boldsymbol{Z}_\star\end{pmatrix}.$$

Equation (4.52) then implies that

$$\begin{aligned}\|\boldsymbol{W}\|_\sigma &\leq \frac{1}{2}\|\boldsymbol{Y}\|_\sigma + \sup_{t\geq 0} e^{t\sigma}\left\|e^{t\mathcal{L}}\boldsymbol{Y}_0\right\|_{C^{1,\gamma}} \leq \frac{1}{2}\|\boldsymbol{Y}\|_\sigma + \sup_{t\geq 0} Ce^{-t(1/4-\sigma)}\|\boldsymbol{Y}_0\|_{C^{1,\gamma}} \\ &\leq \frac{1}{2}\|\boldsymbol{Y}\|_\sigma + C\|\boldsymbol{Y}_0\|_{C^{1,\gamma}} \leq \frac{1}{2}M_{\boldsymbol{Y}} + C\|\boldsymbol{Y}_0\|_{C^{1,\gamma}},\end{aligned} \tag{4.54}$$

where we used Lemma 4.3.2 in the second inequality and the fact that $\sigma < 1/4$ in the third. Equation (4.53) implies

$$\|\boldsymbol{V} - \boldsymbol{Z}_\star\|_0 \leq \frac{1}{2}\|\boldsymbol{Y}\|_\sigma \leq \frac{1}{2}M_{\boldsymbol{Y}}. \tag{4.55}$$

If we thus take $\|\boldsymbol{Y}_0\|_{C^{1,\gamma}} \leq \rho_0 = M_{\boldsymbol{Y}}/2C$ and $M_{\boldsymbol{Y}}/2 \leq M_{\boldsymbol{Z}}$, $\mathcal{S}$ maps $\mathcal{B}_{M_{\boldsymbol{Y}},M_{\boldsymbol{Z}}}$ to itself.

Finally, let $(\boldsymbol{Y}, \boldsymbol{Z})$ be the fixed point of this map. Substituting $\boldsymbol{W} = \boldsymbol{Y}$ and $\boldsymbol{V} = \boldsymbol{Z}$ into (4.54) and (4.55) respectively gives (4.49). □

Proof of item (i) of Theorem 1.2.6. Let $(\boldsymbol{Y}(t), \boldsymbol{Z}(t))$ be the fixed point of the map $\mathcal{S}$ considered in Proposition 4.3.5:

$$\boldsymbol{Y}(t) = e^{t\mathcal{L}}\boldsymbol{Y}_0 + \int_0^t e^{(t-s)\mathcal{L}}\Pi\mathcal{N}(\boldsymbol{Y}(s) + \boldsymbol{Z}(s))ds, \tag{4.56}$$

$$\boldsymbol{Z}(t) = \boldsymbol{Z}_\star + \int_0^t \mathcal{P}\mathcal{N}(\boldsymbol{Y}(s) + \boldsymbol{Z}(s))ds. \tag{4.57}$$

We continue to use the notation used in the proof of Proposition 4.3.5. From (4.49), we see that:

$$\|\boldsymbol{Y}(t)\|_{C^{1,\gamma}} \leq C\|\boldsymbol{Y}_0\|_{C^{1,\gamma}} e^{-\sigma t},\ 0 < \sigma < 1/4. \tag{4.58}$$

We first show that we may replace the above exponential decay with $e^{-t/4}$ with a possible adjustment of the constant C. Take the $C^{1,\gamma}$ norm on both sides of (4.56).

$$\begin{aligned}\|\boldsymbol{Y}(t)\|_{C^{1,\gamma}} &\leq \left\|e^{t\mathcal{L}}\boldsymbol{Y}_0\right\|_{C^{1,\gamma}} + \int_0^t \left\|e^{(t-s)\mathcal{L}}\Pi\mathcal{N}(\boldsymbol{Y}(s) + \boldsymbol{Z}(s))\right\|_{C^{1,\gamma}} ds \\ &\leq C\|\boldsymbol{Y}_0\|_{C^{1,\gamma}} e^{-t/4} + \int_0^t Ce^{-(t-s)/4}\left(\frac{1}{(t-s)^\gamma} + 1\right)\|\mathcal{N}(\boldsymbol{Y}(s) + \boldsymbol{Z}(s))\|_{C^{\lfloor 2\gamma\rfloor, 2\gamma - \lfloor 2\gamma\rfloor}} ds.\end{aligned}$$

where we used Lemma (4.3.2) in the second inequality. Observe that, using Lemma 4.3.4, we have the following estimate:

$$\begin{aligned}\|\mathcal{N}(\boldsymbol{Y}(t) + \boldsymbol{Z}(t))\|_{C^{\lfloor 2\gamma\rfloor, 2\gamma - \lfloor 2\gamma\rfloor}} &= \|\mathcal{N}(\boldsymbol{Y}(t) + \boldsymbol{Z}(t)) - \mathcal{N}(\boldsymbol{Z}(t))\|_{C^{\lfloor 2\gamma\rfloor, 2\gamma - \lfloor 2\gamma\rfloor}} \\ &\leq C\|\boldsymbol{Y}(t)\|_{C^{1,\gamma}}^2 \leq C\|\boldsymbol{Y}_0\|_{C^{1,\gamma}}^2 e^{-2\sigma t},\end{aligned} \tag{4.59}$$

where we used (4.43) in the equality and (4.58) in the last inequality. We thus obtain:

$$\|\boldsymbol{Y}(t)\|_{C^{1,\gamma}} \leq C\|\boldsymbol{Y}_0\|_{C^{1,\gamma}} e^{-t/4} + C\int_0^t e^{-(t-s)/4}e^{-2\sigma s}\left(\frac{1}{(t-s)^\gamma} + 1\right)ds\,\|\boldsymbol{Y}_0\|_{C^{1,\gamma}}^2.$$

If we take $\sigma > 1/8$, for $t \geq 1$,

$$
\begin{aligned}
&\int_0^t e^{-(t-s)/4} e^{-2\sigma s} \left(\frac{1}{(t-s)^\gamma} + 1 \right) ds = e^{-2\sigma t} \int_0^t e^{(2\sigma - 1/4)u} \left(\frac{1}{u^\gamma} + 1 \right) du \\
&\quad \leq e^{-2\sigma t} \left(e^{(2\sigma-1/4)} \int_0^1 \left(\frac{1}{u^\gamma} + 1 \right) du + 2 \int_1^t e^{(2\sigma-1/4)u} du \right) \\
&\quad = e^{-2\sigma t} \left(e^{(2\sigma-1/4)} \left(\frac{1}{1-\gamma} + 1 \right) + \frac{8}{8\sigma - 1} \left(e^{(2\sigma-1/4)t} - 1 \right) \right) \leq C e^{-t/4}.
\end{aligned}
\tag{4.60}
$$

For $t < 1$, the second integral in the second line above is not needed. We thus have,

$$
\|\boldsymbol{Y}(t)\|_{C^{1,\gamma}} \leq C(\|\boldsymbol{Y}_0\|_{C^{1,\gamma}} + \|\boldsymbol{Y}_0\|_{C^{1,\gamma}}^2) e^{-t/4} \leq C(1+\rho_0) \|\boldsymbol{Y}_0\|_{C^{1,\gamma}} e^{-t/4}, \tag{4.61}
$$

where we used the assumption on $\|\boldsymbol{Y}_0\|_{C^{1,\gamma}}$ stated in Proposition 4.3.5 in the last inequality.

We now turn to the component $\boldsymbol{Z}$. Take the norm on both sides of (4.57).

$$
\begin{aligned}
\|\boldsymbol{Z}(t)\|_{C^{1,\gamma}} &\leq \|\boldsymbol{Z}_\star\| + \int_0^t \|\mathcal{N}(\boldsymbol{Y}(s), \boldsymbol{Z}(s))\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} ds \\
&\qquad \leq \|\boldsymbol{Z}_\star\| + C \|\boldsymbol{Y}_0\|_{C^{1,\gamma}}^2 \int_0^t e^{-s/2} ds \leq \|\boldsymbol{Z}_\star\|_{C^{1,\gamma}} + 2C \|\boldsymbol{Y}_0\|_{C^{1,\gamma}}^2,
\end{aligned}
\tag{4.62}
$$

where we used (4.61) in the second inequality. Thus, the following is well-defined.

$$
\boldsymbol{Z}_\infty := \boldsymbol{Z}_\star + \int_0^\infty \mathcal{P}\mathcal{N}(\boldsymbol{Y}(s), \boldsymbol{Z}(s)) ds.
$$

It is clear that $\boldsymbol{Z}_\infty \in \mathcal{V}$ and since it does not degenerate to a point since $|\boldsymbol{Z}(t)|_* \geq m = 1/\pi$ by construction, (see (4.50)). $\boldsymbol{Z}_\infty \in \widehat{\mathcal{V}}$ is a uniformly parametrized circle. We have:

$$
\begin{aligned}
\|\boldsymbol{Z}(t) - \boldsymbol{Z}_\infty\|_{C^{1,\gamma}} &\leq \int_t^\infty \|\mathcal{N}(\boldsymbol{Y}(s), \boldsymbol{Z}(s))\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} ds \\
&\qquad \leq C \|\boldsymbol{Y}_0\|_{C^{1,\gamma}}^2 \int_t^\infty e^{-s/2} ds = 2C \|\boldsymbol{Y}_0\|_{C^{1,\gamma}}^2 e^{-t/2}.
\end{aligned}
\tag{4.63}
$$

Thus, $\boldsymbol{Z}$ converges to $\boldsymbol{Z}_\infty$ exponentially with the above rate.

We must still show that $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}$ is indeed a solution to the Peskin problem. Suppose $\boldsymbol{X}_0 \in h^{1,\gamma}(\mathbb{S}^1)$ is our initial data with $\Pi \boldsymbol{X}_0 = \boldsymbol{Y}_0$ and $\mathcal{P}\boldsymbol{X}_0 = \boldsymbol{Z}_0$. We first assume that $\boldsymbol{Z}_0$ is a uniformly parametrized unit circle centered at the origin, $\boldsymbol{Z}_0 = \boldsymbol{Z}_\star$. This restriction will be later lifted with a scaling argument. From (4.56) and (4.57), by a simple adaptation of the proof of Lemma 4.1.7, we see that $\boldsymbol{Y} \in C([0,T); \mathcal{W}) \cap C^1((0,T); C^\gamma(\mathbb{S}^1))$ for any $T > 0$ ($\boldsymbol{Z}$ lives in a finite dimensional space so its differentiability is automatic) and that $\boldsymbol{Y}$ and $\boldsymbol{Z}$ satisfy the strong form of the equations for $t > 0$

$$\begin{aligned}\partial_t \boldsymbol{Y} &= \mathcal{L}\boldsymbol{Y} + \Pi\mathcal{N}(\boldsymbol{Y}+\boldsymbol{Z}),\\ \partial_t \boldsymbol{Z} &= \mathcal{P}\mathcal{N}(\boldsymbol{Y}+\boldsymbol{Z}),\end{aligned}$$

with $(\boldsymbol{Y}(t), \boldsymbol{Z}(t)) \to (\boldsymbol{Y}_0, \boldsymbol{Z}_0)$ in the $C^{1,\gamma}$ norm. Adding these two equations together and letting $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}$, we see that $\boldsymbol{X}$ satisfies:

$$\begin{aligned}&\partial_t \boldsymbol{X} = \mathcal{L}\boldsymbol{X} + \mathcal{N}(\boldsymbol{X}),\ \boldsymbol{Y} = \Pi\boldsymbol{X},\ \boldsymbol{Z} = \mathcal{P}\boldsymbol{X},\\ &\boldsymbol{X}(t) \to \boldsymbol{X}_0 \text{ in } C^{1,\gamma}(\mathbb{S}^1).\end{aligned}$$

Recall from (4.42) that

$$\mathcal{L}\boldsymbol{X} + \mathcal{N}(\boldsymbol{X}) = \mathcal{R}(\boldsymbol{X}) = \Lambda\boldsymbol{X} + \mathcal{R}(\boldsymbol{X}).$$

We thus see that $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}$ is in fact a strong solution of the Peskin problem, and is consequently also a mild solution of the Peskin problem thanks to Lemma 4.1.8. By the uniqueness result for mild solutions (Theorem 1.2.3), $\boldsymbol{X} = \boldsymbol{Y} + \boldsymbol{Z}$ is *the* mild solution to the Peskin initial value problem with $\boldsymbol{X} = \boldsymbol{X}_0$.

We finally lift the restriction that $\mathcal{P}\boldsymbol{X}_0 = \boldsymbol{Z}_0$ is a uniformly parametrized unit circle centered at the origin. Take any initial data $\boldsymbol{X}_0 \in h^{1,\gamma}(\mathbb{S}^1)$ and its mild solution $\boldsymbol{X}(t)$. Let R be the radius of the circle $\boldsymbol{Z}_0$. Then, if we set:

$$\widehat{\boldsymbol{X}_0} = \frac{1}{R}\left(\boldsymbol{X}_0 - \boldsymbol{p}_{\mathrm{ctr}}\right),\ \widehat{\boldsymbol{X}}(t) = \frac{1}{R}\left(\boldsymbol{X} - \boldsymbol{p}_{\mathrm{ctr}}\right),\quad \boldsymbol{p}_{\mathrm{ctr}} = \mathcal{P}_{\mathrm{trl}}\boldsymbol{X}_0, \tag{4.64}$$

where $\mathcal{P}_{\mathrm{trl}}$ was defined in (4.34) and $\boldsymbol{p}_{\mathrm{ctr}}$ is simply the center point of the circle $\mathcal{P}\boldsymbol{X}_0$. By dilation and translation invariance, $\widehat{\boldsymbol{X}}(t)$ is a solution to the Peskin problem with initial data $\widehat{\boldsymbol{X}_0}$. By design, $\mathcal{P}\widehat{\boldsymbol{X}_0}$ is a uniformly parametrized circle centered at the origin. We may thus apply the results (4.61) and (4.63) to obtain the estimate if $\left\|\widehat{\boldsymbol{Y}}_0\right\|_{C^{1,\gamma}} \leq \rho_0$:

$$\begin{aligned}\left\|\widehat{\boldsymbol{Y}}(t)\right\|_{C^{1,\gamma}} &\leq C\left\|\widehat{\boldsymbol{Y}}_0\right\|_{C^{1,\gamma}} e^{-t/4},\\ \left\|\widehat{\boldsymbol{Z}}(t) - \widehat{\boldsymbol{Z}}_\infty\right\|_{C^{1,\gamma}} &\leq C\left\|\widehat{\boldsymbol{Y}}_0\right\|_{C^{1,\gamma}}^2 e^{-t/2}\end{aligned} \tag{4.65}$$

where $\widehat{\boldsymbol{Y}}(t) = \Pi\widehat{\boldsymbol{X}}(t)$, $\widehat{\boldsymbol{Z}}(t) = \mathcal{P}\widehat{\boldsymbol{X}}(t)$, $\widehat{\boldsymbol{Y}}_0 = \Pi\widehat{\boldsymbol{X}_0}$, $\widehat{\boldsymbol{Z}}_0 = \mathcal{P}\widehat{\boldsymbol{X}_0}$ and $\widehat{\boldsymbol{Z}}_\infty$ is the point to which $\widehat{\boldsymbol{Z}}(t)$ converges (whose existence was guaranteed above). From (4.64), we see that

$$\widehat{\boldsymbol{Y}}(t) = \frac{1}{R}\boldsymbol{Y}(t),\ \widehat{\boldsymbol{Y}}_0 = \frac{1}{R}\boldsymbol{Y}_0,\ \widehat{\boldsymbol{Z}}(t) = \frac{1}{R}\left(\boldsymbol{Z}(t) - \boldsymbol{p}_{\mathrm{ctr}}\right).$$

By plugging in the above into (4.65), we obtain the inequalities (1.29) and (1.30) by setting $\boldsymbol{Z}_\infty = R\widehat{\boldsymbol{Z}}_\infty + \boldsymbol{p}_{\text{ctr}}$.

Finally, note that

$$\|\boldsymbol{X} - \boldsymbol{Z}_\infty\|_{C^{1,\gamma}} = \|\Pi\boldsymbol{X} + \mathcal{P}\boldsymbol{X} - \boldsymbol{Z}_\infty\|_{C^{1,\gamma}} \leq \|\Pi\boldsymbol{X}\|_{C^{1,\gamma}} + \|\mathcal{P}\boldsymbol{X} - \boldsymbol{Z}_\infty\|_{C^{1,\gamma}}.$$

Inequality (1.33) is a direct consequence of this and (1.29) and (1.30). □

To obtain exponential decay in higher Hölder norms, we state a result that is a direct consequence of Lemma 4.2.3 and the proof of Proposition 4.2.4. We omit the proof since it will be an almost exact reiteration of the proof of Proposition 4.2.4.

Lemma 4.3.6. *Suppose $\boldsymbol{X}(t)$ is a mild solution to the Peskin problem up to $t = \epsilon$ satisfying*

$$\|\boldsymbol{X}(t)\|_{C^{1,\gamma}} \leq M \text{ and } |\boldsymbol{X}(t)|_* \geq m \text{ for } 0 \leq t \leq \epsilon.$$

Then, for any $n \in \mathbb{N}$ and $0 < \alpha < 1$, we have:

$$\|\boldsymbol{X}(\epsilon)\|_{C^{n,\alpha}} \leq M_{n,\alpha}$$

where $M_{n,\alpha}$ is a constant that depends only on $n, \alpha, \epsilon, \gamma, M$ and m.

Proof of item (ii) of Theorem 1.2.6. Like the above proof of item (i) in Theorem 1.2.6, we first assume that $\mathcal{P}\boldsymbol{X}_0$ is a unit circle centered at the origin and later use a scaling argument to obtain the result for general initial data. First, recall from (4.50) that

$$\sup_{t\geq 0} \|\boldsymbol{X}(t)\|_{C^{1,\gamma}} = \|\boldsymbol{Z}_0\|_{C^{1,\gamma}} + \frac{1}{\pi} < \infty,$$
$$\inf_{t\geq 0} |\boldsymbol{X}(t)|_* \geq \frac{1}{\pi} > 0.$$

Applying Lemma (4.3.6) to the solution $\boldsymbol{X}(t+\tau), t \geq 0$ ($\tau \geq 0$ considered a parameter),

$$\|\boldsymbol{X}(\tau + \epsilon)\|_{C^{n,\alpha}} \leq M_{n,\alpha} \text{ for any } \tau \geq 0,$$

where M_n depends only on n, α, ϵ and γ. Thus,

$$\sup_{\tau\geq\epsilon} \|\boldsymbol{X}(t)\|_{C^{n,\alpha}} \leq M_{n,\alpha}, \; n \in \mathbb{N}, 0 < \alpha < 1.$$

Lemma 4.2.3 implies that:

$$\sup_{\tau \geq \epsilon} \|\mathcal{R}(\boldsymbol{X}(\tau))\|_{C^{n,\alpha}} \leq M^{\mathcal{R}}_{n,\alpha},\ n \in \mathbb{N}, 0 < \alpha < 1.$$

where the constant $M^{\mathcal{R}}_{n,\alpha}$ again depends only on n, α, ϵ and γ. Recall that:

$$\mathcal{N}(\boldsymbol{X}) = -\mathcal{Q}\boldsymbol{X} + \mathcal{R}(\boldsymbol{X}),\ \mathcal{Q}\boldsymbol{X} = \mathcal{L}\boldsymbol{X} - \Lambda\boldsymbol{X}.$$

Using the fact that $\mathcal{Q}$ is a bounded operator from $C^{n,\alpha}(\mathbb{S}^1)$ to itself, (see (4.38); the Hilbert transform is bounded from $C^{n,\alpha}$ to itself as long as $0 < \alpha < 1$), we have:

$$\|\mathcal{N}(\boldsymbol{X}(t))\|_{C^{n,\alpha}} \leq M^{\mathcal{N}}_{n,\alpha} \text{ for } t \geq \epsilon. \tag{4.66}$$

where the above constant depends only on n, α, ϵ and γ. On the other hand, combining (4.59) and (4.61), we have:

$$\|\mathcal{N}(\boldsymbol{X}(t))\|_{C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor}} \leq C \|\boldsymbol{Y}_0\|^2_{C^{1,\gamma}} e^{-t/2}, \tag{4.67}$$

Interpolating the two estimates (4.66) and (4.67), for any $0 < \sigma < 1/4$, we obtain (see Chapter 1 of [21] for results on interpolation in Hölder spaces):

$$\|\mathcal{N}(\boldsymbol{X}(t))\|_{C^{\lfloor \beta \rfloor, \beta - \lfloor \beta \rfloor}} \leq C \|\boldsymbol{Y}_0\|^{8\sigma}_{C^{1,\gamma}} e^{-2\sigma t},\ \text{for } t \geq \epsilon,\ \beta = 8\gamma\sigma + (n+\alpha)(1-4\sigma) \tag{4.68}$$

where the above constant C depends only on $n, \alpha, \epsilon, \gamma$ and σ. Since n can be made arbitrarily large, the above estimate is true for any $\beta \geq 1 + \gamma$ and $0 < \sigma < 1/4$.

Suppose we know that:

$$\|\boldsymbol{Y}(t)\|_{C^{\lfloor \beta \rfloor, \beta - \lfloor \beta \rfloor}} \leq C \|\boldsymbol{Y}_0\|_{C^{1,\gamma}} e^{-t/4},\ t \geq k\epsilon,\ \ \text{for some } k \in \mathbb{N} \tag{4.69}$$

where the above constant C that depends only on β, γ and $k\epsilon$. The above is true if $\beta = 1 + \gamma$ and $k = 1$ by (4.61). Let $\beta < \kappa < 1 + \beta$ and take the $C^{\lfloor \kappa \rfloor, \kappa - \lfloor \kappa \rfloor}$ norm on both sides of (4.56).

We have:

$$
\begin{aligned}
\|\boldsymbol{Y}(t+k\epsilon)\|_{C^{\lfloor\kappa\rfloor,\kappa-\lfloor\kappa\rfloor}} &\leq \left\|e^{t\mathcal{L}}\boldsymbol{Y}(k\epsilon)\right\|_{C^{\lfloor\kappa\rfloor,\kappa-\lfloor\kappa\rfloor}} \\
&+ \int_0^t \left\|e^{(t-s)\mathcal{L}}\Pi\mathcal{N}(\boldsymbol{X}(s+k\epsilon))\right\|_{C^{\lfloor\kappa\rfloor,\kappa-\lfloor\kappa\rfloor}} ds \\
&\leq \frac{C}{t^{\kappa-\beta}}e^{-t/4}\|\boldsymbol{Y}(k\epsilon)\|_{C^{\lfloor\beta\rfloor,\beta-\lfloor\beta\rfloor}} \\
&+ \int_0^t Ce^{-t/4}\left(\frac{1}{(t-s)^{\kappa-\beta}}+1\right)\|\mathcal{N}(\boldsymbol{X}(s+k\epsilon))\|_{C^{\lfloor\beta\rfloor,\beta-\lfloor\beta\rfloor}} ds \\
&\leq \frac{C}{t^{\kappa-\beta}}e^{-t/4}\|\boldsymbol{Y}_0\|_{C^{1,\gamma}} \\
&+ \int_0^t Ce^{-(t-s)/4}\left(\frac{1}{(t-s)^{\kappa-\beta}}+1\right)\|\boldsymbol{Y}_0\|_{C^{1,\gamma}}^{8\sigma} e^{-2\sigma t}ds,
\end{aligned}
$$

where we used Lemma 4.3.2 in the second inequality and (4.68) and (4.69) in the last inequality. Letting $\sigma > 1/8$, we see from (4.60) that:

$$
\begin{aligned}
\|\boldsymbol{Y}(t+k\epsilon)\|_{C^{\lfloor\kappa\rfloor,\kappa-\lfloor\kappa\rfloor}} &\leq C\left(\frac{1}{t^{\kappa-\beta}}\|\boldsymbol{Y}_0\|_{C^{1,\gamma}}+\|\boldsymbol{Y}_0\|_{C^{1,\gamma}}^{8\sigma}\right)e^{-t/4} \\
&\leq C\left(\frac{1}{\epsilon^{\kappa-\beta}}+\rho_0^{8\sigma-1}\right)\|\boldsymbol{Y}_0\|_{C^{1,\gamma}}e^{-t/4},
\end{aligned}
$$

where we used the assumption that $\|\boldsymbol{Y}_0\| \leq \rho_0$ (see statement of Proposition 4.3.5) and $8\sigma-1 > 0$ in the last inequality. Therefore,

$$
\|\boldsymbol{Y}(t)\|_{C^{\lfloor\kappa\rfloor,\kappa-\lfloor\kappa\rfloor}} \leq C\|\boldsymbol{Y}_0\|_{C^{1,\gamma}}e^{-t/4},\ t \geq (k+1)\epsilon,
$$

where the above constant C that depends only on κ, γ and $(k+1)\epsilon$. Starting with $\beta = 1+\gamma$, we may iterate this process indefinitely to find that the above estimate is true for any κ with a suitably large k. Since ϵ can be taken arbitrarily small and k is finite, we may replace $k\epsilon$ with ϵ in the above by making the constant C larger if necessary, to obtain:

$$
\|\boldsymbol{Y}(t)\|_{C^{\lfloor\kappa\rfloor,\kappa-\lfloor\kappa\rfloor}} \leq C\|\boldsymbol{Y}_0\|_{C^{1,\gamma}}e^{-t/4},\ t \geq \epsilon, \tag{4.70}
$$

where C depends only on γ, κ and ϵ. To obtain the same bound for the C^n norm, as stated in (1.32), simply note that the $C^{n,\alpha}$ norm for $0 < \alpha < 1$ dominates the C^n norm.

Finally, to obtain (1.32) for general initial conditions, we may use the same scaling argument used at the end of the proof of item (i) of Theorem 1.2.6 above. Note also that

$$
\|\boldsymbol{X}-\boldsymbol{Z}_\infty\|_{C^{n,\gamma}} \leq \|\Pi\boldsymbol{X}\|_{C^{n,\gamma}}+\|\mathcal{P}\boldsymbol{X}-\boldsymbol{Z}_\infty\|_{C^{n,\gamma}} \leq \|\Pi\boldsymbol{X}\|_{C^{n,\gamma}}+C\|\mathcal{P}\boldsymbol{X}-\boldsymbol{Z}_\infty\|_{\mathcal{V}}
$$

where the last inequality follows from the equivalence of all norms in finite dimensional spaces. Thus, (1.33) is thus a direct consequence of (1.30) and (1.32). $\square$

4.3.4 Computational Verification

Here, we computationally verify the exponential decay rate to the circle. We developed a numerical scheme to simulate the Peskin problem based on the small scale decomposition (1.18). The numerical scheme is second order accurate in time t and spectrally accurate in θ. We point out that the second order accuracy in time (as opposed to a first-order scheme) turned out to be crucial in computationally verifying the asymptotic decay rate.

We first give a description of the numerical scheme. We use equation (1.18) and (1.19) as the basis for our algorithm. Discretize $\mathbb{S}^1$ with N points so that $Nh = 2\pi$, where h is the grid spacing. Let $\theta = \theta_k = k\Delta\theta, k = 0, 1, \cdots N-1$ be the grid locations, and $\boldsymbol{X}_{h,k} = (X_{h,k}, Y_{h,k})$ be the numerically approximated value of $\boldsymbol{X}(\theta_k)$. We let N be even. For a function W defined on the discrete θ grid, define $\mathcal{F}_h$ to be the discrete Fourier transform:

$$(\mathcal{F}_h W)_k = \sum_{l=0}^{N-1} \exp(-2\pi i k l/N) W_l,\ k = -N/2+1, \cdots, -1, 0, 1, \cdots N/2.$$

Define the approximation to the derivative $\mathcal{D}_h$ and the semigroup $\mathcal{S}_h(t)$ as follows:

$$\mathcal{D}_h W = \mathcal{F}_h^{-1} \widehat{\mathcal{D}}_h(k) \mathcal{F}_h W,\ \widehat{\mathcal{D}}_h(k) = \begin{cases} ik & \text{if } k \neq N/2, \\ 0 & \text{if } k = N/2, \end{cases}$$

$$\mathcal{S}_h(t) W = \mathcal{F}_h^{-1} \widehat{\mathcal{S}}_h(t,k) \mathcal{F}_h W,\ \widehat{\mathcal{S}}_h(t,k) = \begin{cases} e^{-t|k|/4} & \text{if } k \neq N/2, \\ 0 & \text{if } k = N/2. \end{cases}$$

Recall from (1.18) that $\mathcal{R}$ can be written as:

$$\mathcal{R} = -\frac{1}{4\pi} \int_{\mathbb{S}^1} \partial'_\theta \left(-\log\left(\frac{|\Delta \boldsymbol{X}|}{2\,|\sin((\theta-\theta')/2)|} \right) I + \frac{\Delta \boldsymbol{X} \otimes \Delta \boldsymbol{X}}{|\Delta \boldsymbol{X}|^2} \right) \partial'_\theta \boldsymbol{X}' d\theta'.$$

We now approximate $\mathcal{R}$. Define:

$$H_{kl} = -\log\left(\frac{|\boldsymbol{X}_{h,k} - \boldsymbol{X}_{h,l}|}{2\,|\sin((\theta_k-\theta_l)/2)|} \right) I + \frac{(\boldsymbol{X}_{h,k} - \boldsymbol{X}_{h,l}) \otimes (\boldsymbol{X}_{h,k} - \boldsymbol{X}_{h,l})}{|\boldsymbol{X}_{h,k} - \boldsymbol{X}_{h,l}|^2} \text{ for } l \neq k,$$

$$H_{kk} = -\log|(\mathcal{D}_h \boldsymbol{X}_h)_k|\, I + \frac{(\mathcal{D}_h \boldsymbol{X}_h)_k \otimes (\mathcal{D}_h \boldsymbol{X}_h)_k}{|(\mathcal{D}_h \boldsymbol{X}_h)_k|^2}.$$

We let the approximation of $\mathcal{R}$ at $\theta = \theta_k$ be:

$$\mathcal{R}_{h,k}(\boldsymbol{X}_h) = -\frac{1}{4\pi} \sum_{l=0}^{N-1} (\mathcal{D}_{h,j} H_{kj})_l (\mathcal{D}_h \boldsymbol{X}_h)_l h$$

where $\mathcal{D}_{h,j}$ means that the operator $\mathcal{D}_h$ is acting on the grid function with argument j.

Let $t_n = n\triangle t$ where $\triangle t$ is the time step. We use the following Runge-Kutta type approximation scheme. Before we describe our time-stepping scheme, note from (1.19) that:

$$\begin{aligned}\boldsymbol{X}(t_{n+1}) =& e^{\triangle t\Lambda}\boldsymbol{X}(t_n) + \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-s)\Lambda}\mathcal{R}(\boldsymbol{X}(s))ds \\ \approx& e^{\triangle t\Lambda}\boldsymbol{X}(t_n) + e^{\triangle t\Lambda/2}\mathcal{R}(\boldsymbol{X}(t_{n+1/2}))\triangle t, \quad t_{n+1/2} = t_n + \triangle t/2.\end{aligned}$$

In order to approximate $\boldsymbol{X}(t_{n+1/2})$, we just use:

$$\boldsymbol{X}(t_{n+1/2}) \approx e^{\triangle t\Lambda/2}\boldsymbol{X}(t_n) + e^{\triangle t\Lambda/2}\mathcal{R}(\boldsymbol{X}(t_n))\triangle t/2.$$

We use the spatially discrete version of the above for our time-stepping. Let $\boldsymbol{X}_h^n$ be the discrete approximation to $\boldsymbol{X}$ at time $t = t_n = n\triangle t$. We let:

$$\begin{aligned}\boldsymbol{X}_h^{n+1/2} &= \mathcal{S}_h(\triangle t/2)\left(\boldsymbol{X}_h^n + \mathcal{R}_h(\boldsymbol{X}_h^n)\triangle t/2\right), \\ \boldsymbol{X}_h^{n+1} &= \mathcal{S}_h(\triangle t)\boldsymbol{X}_h^n + \mathcal{S}_h(\triangle t/2)\mathcal{R}_h(\boldsymbol{X}_h^{n+1/2})\triangle t.\end{aligned}$$

This concludes our description of the numerical scheme.

To define the discrete projection operators, define the following discrete inner product for grid functions $\boldsymbol{V}$ and $\boldsymbol{W}$:

$$\langle \boldsymbol{V}, \boldsymbol{W}\rangle_h = \sum_{k=0}^{N-1}(\boldsymbol{V}_k \cdot \boldsymbol{W}_k)h.$$

Let $\boldsymbol{e}_{x,h}, \boldsymbol{e}_{y,h}, \boldsymbol{e}_{\mathrm{r},h}$ and $\boldsymbol{e}_{\mathrm{t},h}$ simply be the evaluation of the vectors $\boldsymbol{e}_{x,y}, \boldsymbol{e}_{\mathrm{r,t}}$ in (1.26) evaluated at the grid points θ_k. Define the discrete projection operators:

$$\begin{aligned}\mathcal{P}_h\boldsymbol{V} &= \frac{1}{2\pi}\left(\langle \boldsymbol{V}, \boldsymbol{e}_{x,h}\rangle_h \boldsymbol{e}_{x,h} + \langle \boldsymbol{V}, \boldsymbol{e}_{y,h}\rangle_h \boldsymbol{e}_{y,h} + \langle \boldsymbol{V}, \boldsymbol{e}_{\mathrm{r},h}\rangle_h \boldsymbol{e}_{\mathrm{r},h} + \langle \boldsymbol{V}, \boldsymbol{e}_{\mathrm{t},h}\rangle_h \boldsymbol{e}_{\mathrm{t},h}\right) \\ \Pi_h\boldsymbol{V} &= \boldsymbol{V} - \mathcal{P}_h\boldsymbol{V}.\end{aligned}$$

Let us also define the discrete C^1 norm as follows:

$$\|\boldsymbol{V}\|_{C_h^1} = \sup_k |\boldsymbol{V}_k| + \sup_k |(\mathcal{D}_h\boldsymbol{V})_k|.$$

To numerically compute the decay rate to the circle, we take four different initial conditions. They are:

$$\boldsymbol{X}_0 = \begin{pmatrix}\cos(\theta) + \cos(2\theta)/5 - \sin(2\theta)/10 \\ \sin(\theta) + \sin(2\theta)/5 + \cos(2\theta)/10\end{pmatrix} \tag{4.71}$$

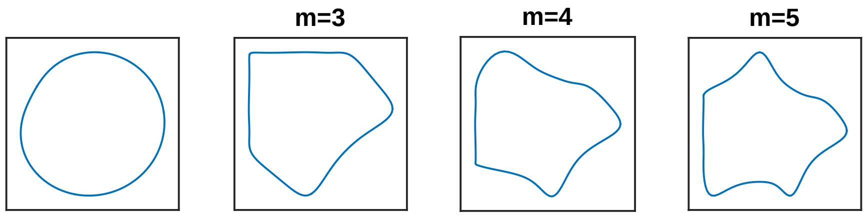

Figure 4.1: Initial Curves. The unlabeled curve corresponds to equation (4.71), where as the other three correspond to $m = 3, 4, 5$ respectively in equation (4.72).

and

$$\boldsymbol{X}_0 = \begin{pmatrix} (1 + \exp(\cos(3\theta))/4)\cos(\theta) \\ (1 + \exp(\sin(m\theta))/4)\sin(\theta) \end{pmatrix}, \quad m = 3, 4, 5. \tag{4.72}$$

The configurations of the above initial curves can be found in Figure 4.1. The initial curve (4.71) corresponds to a perturbation of the unit circle proportional to the primary decay mode (4.37).

We simulated the dynamics with the above initial data with $N = 128$ and $\triangle t = 0.01$. Let us recall from Theorem 1.2.6 that $\Pi\boldsymbol{X}$ decays to 0 at an exponential rate of $e^{-t/4}$ and $\mathcal{P}\boldsymbol{X}$ decays to some uniformly parametrized circle at an exponential rate of $e^{-t/2}$. To computationally verify the decay result for $\Pi\boldsymbol{X}$, we compute

$$\|\Pi_h \boldsymbol{X}_h^n\|_{C_h^1}. \tag{4.73}$$

For $\mathcal{P}\boldsymbol{X}$, the circle to which $\boldsymbol{X}$ converges is unknown and thus we instead compute the decay of the time derivative. For $\mathcal{P}\boldsymbol{X}$, we have:

$$\mathcal{P}\boldsymbol{X}(t) = a_x(t)\boldsymbol{e}_x + a_y(t)\boldsymbol{e}_y + a_{\mathrm{r}}(t)\boldsymbol{e}_{\mathrm{r}} + a_{\mathrm{t}}(t)\boldsymbol{e}_{\mathrm{t}}.$$

The proof of Theorem 1.2.6 implies that

$$|d\boldsymbol{a}/dt|, \ \boldsymbol{a} = (a_x, a_y, a_{\mathrm{r}}, a_{\mathrm{t}})^{\mathrm{T}} \text{ decays like } e^{-t/2},$$

where the absolute value above is the Euclidean norm in $\mathbb{R}^4$. Let:

$$\mathcal{P}_h \boldsymbol{X}_h^n = a_{x,h}^n \boldsymbol{e}_{x,h} + a_{y,h}^n \boldsymbol{e}_{y,h} + a_{\mathrm{r},h}^n \boldsymbol{e}_{\mathrm{r},h} + a_{\mathrm{t},h}^n \boldsymbol{e}_{\mathrm{t},h}.$$

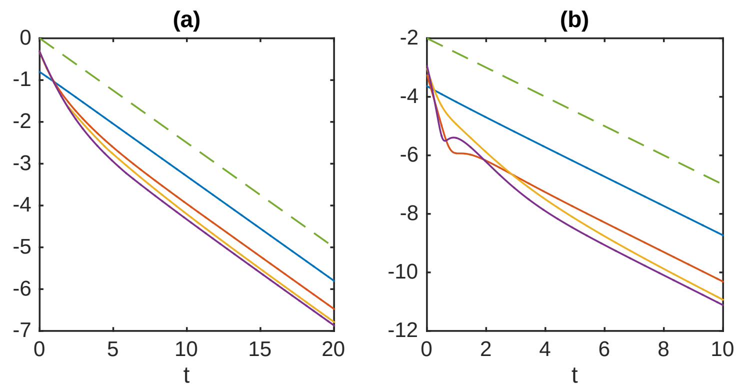

Figure 4.2: Decay Rates. In Figure (a), $\log(\|\Pi_h \boldsymbol{X}_h^n\|_{C_h^1})$ is plotted against $t = n\triangle t$ for the four different initial data given in (4.71) and (4.72). The dashed line has a slope $-1/4$, indicating the theoretical decay rate. In Figure (b), $\log(\left|(\mathcal{D}_t \boldsymbol{a}_h)^{n+1/2}\right|)$ is plotted against $t = (n+1/2)\triangle t$ with the four different initial data. The dashed line has a slope of $-1/2$, the theoretical decay rate. The solid curves that are almost straight in both figures correspond to the initial data (4.71). Note that we have plotted $\log(\|\Pi_h \boldsymbol{X}_h^n\|_{C_h^1})$ up to $t = 20$ whereas $\log(\left|(\mathcal{D}_t \boldsymbol{a}_h)^{n+1/2}\right|)$ is plotted up to $t = 10$, in accordance with the fact that the latter decays twice as fast as the former.

Set:

$$(\mathcal{D}_t \boldsymbol{a}_h)^{n+1/2} = \frac{\boldsymbol{a}_h^{n+1} - \boldsymbol{a}_h^n}{\triangle t}, \ \boldsymbol{a}_h = (a_{x,h}, a_{y,h}, a_{\mathrm{r},h}, a_{\mathrm{t},h})^{\mathrm{T}}.$$

We thus compute

$$\left|(\mathcal{D}_t \boldsymbol{a}_h)^{n+1/2}\right|. \tag{4.74}$$

The norms (4.73) and (4.74) are plotted in Figure 4.2. It is clearly seen that the asymptotic decay rate conforms to the theory. The decay with initial data (4.71) is almost precisely exponential at the theoretical rate, which is to be expected given that the perturbation is proportional to the primary decay modes in (4.37). For the other initial data, the decay rate asymptotically approaches the theoretically predicted value.

4.4 Global Behavior

Recall that the area and energy identities (1.14) and (1.24) are satisfied for mild solutions, as proved in Proposition 1.2.5. Take area conservation. Viewing the interior fluid area $|\Omega_{\rm i}|$ as a function of t, we have:

$$|\Omega_{\rm i}|\,(t) = |\Omega|_{\rm i}\,(\epsilon) \text{ for } 0 < \epsilon < t$$

so long as the solution exists up to time t. Given the expression for $|\Omega_{\rm i}|$ given in (1.14), it is clear that the $|\Omega_{\rm i}|$ is a continuous functional of $\boldsymbol{X} \in C^{1,\gamma}(\mathbb{S}^1)$. Since our mild solution $\boldsymbol{X}(t) \in C([0,T]; C^{1,\gamma}(\mathbb{S}^1))$, we may take the limit as $\epsilon \to 0$ in the above equation to find that:

$$|\Omega_{\rm i}|\,(t) = |\Omega_{\rm i}|\,(0).$$

Let us now turn to energy conservation. From (1.24), we have:

$$\mathcal{E}(t) - \mathcal{E}(\epsilon) = -\int_\epsilon^t \mathcal{D}(s)ds \text{ for } 0 < \epsilon < t.$$

The energy functional $\mathcal{E}$ is continuous with respect to $\boldsymbol{X} \in C^{1,\gamma}(\mathbb{S}^1)$, and therefore, we may take the limit as $\epsilon \to 0$ in the above to find:

$$\mathcal{E}(t) - \mathcal{E}(0) = -\int_0^t \mathcal{D}(s)ds. \tag{4.75}$$

so long as the solution exists up to time t. Note that the right hand side does not need to be interpreted as an improper integral since the integrand is non-negative.

We now state a simple observation that is a consequence of the above.

Lemma 4.4.1. *Suppose we have a mild solution $\boldsymbol{X}(t) \in C([0,T]; C^{1,\gamma}(\mathbb{S}^1)), 0 < \gamma < 1$. Then, $|\boldsymbol{X}|_*$ has an upper bound and $\|\boldsymbol{X}\|_{\dot{C}^{1,\gamma}}$ has a lower bound. More concretely,*

$$|\boldsymbol{X}(t)|_* \leq \sqrt{\frac{\mathcal{E}(0)}{\pi}}, \tag{4.76}$$

$$\|\partial_\theta \boldsymbol{X}(t)\|_{C^{0,\gamma}} \geq \sqrt{\frac{|\Omega_{\rm i}(0)|}{\pi}}. \tag{4.77}$$

Proof. We first consider the (4.76). From (4.75), $\mathcal{E}(t) \leq \mathcal{E}(0)$, and therefore,

$$\mathcal{E}(0) \geq \mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{S}^1} |\partial_\theta \boldsymbol{X}|^2 \, d\theta \geq \pi \inf_{\theta\in\mathbb{S}^1} |\partial_\theta \boldsymbol{X}|^2 \geq \pi\, |\boldsymbol{X}|_*^2\,.$$

For (4.77), we use the isoperimetric inequality:

$$\sqrt{4\pi|\Omega_{\mathrm{i}}(0)|} \leq \int_{\mathbb{S}^1} |\partial_\theta \boldsymbol{X}|\, d\theta \leq 2\pi \sup_{\theta\in\mathbb{S}^1} |\partial_\theta \boldsymbol{X}| \leq 2\pi \left\|\partial_\theta \boldsymbol{X}\right\|_{C^{0,\gamma}}.$$

□

We are now ready to prove Theorem 1.2.8.

Proof of Theorem 1.2.8. We first make the following observation. Suppose we have:

$$\sup_{0\leq t<\tau_{\max}(\boldsymbol{X}_0)} \varrho_\alpha(\boldsymbol{X}(t)) = K < \infty. \tag{4.78}$$

By Lemma 4.4.1, we immediately have:

$$\sqrt{\frac{|\Omega_{\mathrm{i}}(0)|}{\pi}} \leq \left\|\partial_\theta \boldsymbol{X}\right\|_{C^{0,\alpha}} \leq K\left|\boldsymbol{X}\right|_* \leq K\sqrt{\frac{\mathcal{E}(0)}{\pi}}. \tag{4.79}$$

A bound on the ratio thus immediately results in upper and lower bounds for $\left\|\partial_\theta \boldsymbol{X}\right\|_{C^{0,\alpha}}$ and $\left|\boldsymbol{X}\right|_*$.

We first consider item (i). Suppose that, for some $\alpha > 0$,

$$\limsup_{t\to\tau_{\max}(\boldsymbol{X}_0)} \varrho_\alpha(\boldsymbol{X}(t)) < \infty.$$

We then show we can then extend the mild solution beyond $\tau_{\max}(\boldsymbol{X}_0)$. Since $\varrho_\alpha > \varrho'_\alpha$ for any $\alpha > \alpha'$, we may as well assume that $0 < \alpha \leq \gamma$ and $\alpha < 1/2$. Then, (4.78) holds, and thus we have (4.79). In particular, we have:

$$m = \frac{1}{K}\sqrt{\frac{|\Omega_{\mathrm{i}}(0)|}{\pi}} \leq \left|\boldsymbol{X}\right|_* \text{ for } 0 \leq t < \tau_{\max}(\boldsymbol{X}_0). \tag{4.80}$$

By (2.60) and (4.6), this implies that there is a constant $M_{\mathcal{R}}$ such that:

$$\left\|\mathcal{R}(\boldsymbol{X}(t))\right\|_{C^{0,2\alpha}} \leq C\frac{\left\|\partial_\theta \boldsymbol{X}(t)\right\|^4_{C^{0,\alpha}}}{\left|\boldsymbol{X}(t)\right|^3_*} \leq M_{\mathcal{R}} \text{ for } 0 \leq t < \tau_{\max}(\boldsymbol{X}_0).$$

Thus,

$$\begin{aligned}
\left\|\boldsymbol{X}(t)\right\|_{C^{1,\alpha}} &\leq C\left\|\boldsymbol{X}_0\right\|_{C^{1,\alpha}} + \int_0^t \left\|e^{(t-s)\Lambda}\mathcal{R}(\boldsymbol{X}(s))\right\| ds \\
&\leq C\left\|\boldsymbol{X}_0\right\|_{C^{1,\alpha}} + \int_0^t \frac{C}{(t-s)^{1-\alpha}} M_{\mathcal{R}} ds \leq M \text{ for } 0 \leq t < \tau_{\max}(\boldsymbol{X}_0).
\end{aligned}$$

From this, we know from Lemma 4.3.6 that, for each $0 < \beta < 1$ there are constants $M_\beta > 0$ such that

$$\|\boldsymbol{X}(t)\|_{C^{2,\beta}} \le M_\beta \text{ for all } \frac{1}{2}\tau_{\max}(\boldsymbol{X}_0) \le t < \tau_{\max}(\boldsymbol{X}_0). \tag{4.81}$$

From the strong form of our equation and Lemma 4.2.3 we see that, for some constant $\widetilde{M}_\beta$,

$$\|\partial_t \boldsymbol{X}\|_{C^{1,\beta}} \le \widetilde{M}_\beta \text{ for all } \frac{1}{2}\tau_{\max}(\boldsymbol{X}_0) \le t < \tau_{\max}(\boldsymbol{X}_0).$$

Applying this to $\beta = \alpha$ in particular, we see that $\boldsymbol{X}(t)$ is uniformly bounded in $C^1([\tau_{\max}(\boldsymbol{X}_0)/2, \tau_{\max}(\boldsymbol{X}_0)), C^{1,\alpha}(\mathbb{S}^1))$. This implies that $\boldsymbol{X}(t) \in C([0, \tau_{\max}(\boldsymbol{X}_0)); C^{1,\alpha}(\mathbb{S}^1))$ is uniformly continuous in time. Thus, the following limit exists in $C^{1,\alpha}(\mathbb{S}^1)$:

$$\lim_{t \to \tau_{\max}(\boldsymbol{X}_0)} \boldsymbol{X}(t) = \boldsymbol{X}_\star.$$

By (4.81), $\boldsymbol{X}_\star \in C^{2,\beta}(\mathbb{S}^1) \subset h^{1,\alpha}(\mathbb{S}^1)$, and by (4.80), we have $|\boldsymbol{X}_\star|_* \ge m > 0$. This means that we may continue the solution on from $\tau_{\max}(\boldsymbol{X}_0)$ using our local existence theorem Theorem 1.2.3. A mild solution in $C^{1,\alpha}(\mathbb{S}^1)$ is a mild solution in $C^{1,\gamma}(\mathbb{S}^1)$ by our regularity results established in Theorem 1.2.4.

Let us next consider item (ii). Our method is to consider the ω-limit set of the global solution with bounded deformation ratio, although we will not explicitly use the terminology associated with ω-limit sets (see [31] for example) since our setting is quite simple. Define:

$$\widehat{\boldsymbol{X}} = \Pi_{\mathrm{trl}} \boldsymbol{X},$$

where Π_{trl} is the projection operator defined in (4.34). Clearly,

$$\left\|\partial_\theta \widehat{\boldsymbol{X}}\right\|_{C^{0,\beta}} = \|\partial_\theta \boldsymbol{X}\|_{C^{0,\beta}}, \; 0 < \beta < 1.$$

Furthermore, it is easily seen that the following Poincaré type inequality holds

$$\left\|\widehat{\boldsymbol{X}}\right\|_{C^{1,\beta}} \le C \left\|\partial_\theta \widehat{\boldsymbol{X}}\right\|_{C^{0,\beta}}, \tag{4.82}$$

for some constant C independent of $\boldsymbol{X}$. Indeed, let $\widehat{\boldsymbol{X}} = (\widehat{X}, \widehat{Y})$. By the definition of Π_{trl}, we have:

$$\int_{\mathbb{S}^1} \widehat{X} d\theta = 0.$$

There must thus be a point $\theta_* \in \mathbb{S}^1$ at which $\widehat{X}(\theta_*) = 0$. Thus,

$$\left|\widehat{X}(\theta)\right| \leq \int_{\theta_*}^{\theta} \left|\partial_\theta \widehat{X}\right| d\theta \leq 2\pi \left\|\partial_\theta \widehat{X}\right\|_{C^0}.$$

A similar bound can be found for $\widehat{Y}$.

Suppose bound (4.78) is satisfied with $\tau_{\max} = \infty$. By (4.79) and (4.82), this implies that:

$$\sup_{t \geq 0} \left\|\widehat{\boldsymbol{X}}\right\|_{C^{1,\alpha}} \equiv M_\alpha < \infty. \tag{4.83}$$

Take any sequence $t_1 < t_2 < t_3 < \cdots t_k \to \infty$ and let $\boldsymbol{X}_k = \boldsymbol{X}(t_k)$. The set consisting of $\widehat{\boldsymbol{X}_k} = \Pi_{\mathrm{trl}} \boldsymbol{X}_k$ is precompact in $C^{1,\beta}(\mathbb{S}^1), 0 < \beta < \alpha$, thanks to (4.83). We may thus extract a subseqence of the time points above, which we shall continue to call $t = t_k$, so that $\widehat{\boldsymbol{X}_k} \to \widehat{\boldsymbol{X}_\star}$ in $C^{1,\beta}(\mathbb{S}^1)$ for some $\widehat{\boldsymbol{X}_\star} \in C^{1,\beta}(\mathbb{S}^1)$. In fact, $\widehat{\boldsymbol{X}_\star} \in h^{1,\beta}(\mathbb{S}^1)$ since it is in the completion of $C^{1,\alpha}(\mathbb{S}^1)$ in $C^{1,\beta}(\mathbb{S}^1)$.

Consider the mild solutions $\boldsymbol{W}_k(t)$ and $\boldsymbol{W}_\star(t)$ with initial data $\widehat{\boldsymbol{X}_k}$ and $\widehat{\boldsymbol{X}_\star}$ respectively. Note that there is a local mild solution with initial data $\widehat{\boldsymbol{X}_\star}$ thanks to (4.79); $\left|\widehat{\boldsymbol{X}_\star}\right|_*$ is bounded from below. By continuity with respect to initial data established in Theorem 1.2.3, there exists a $T > 0$ such that, for k sufficiently large, $\boldsymbol{W}_k(t)$ is well-defined for $0 \leq t \leq T$, and

$$\boldsymbol{W}_k(t) \to \boldsymbol{W}_\star(t) \text{ in } C([0,T]; C^{1,\beta}(\mathbb{S}^1)). \tag{4.84}$$

We now argue that the energy $\mathcal{E}$ is constant on the solution $\boldsymbol{W}_\star(t)$. Let us use the notation $\mathcal{E}(\boldsymbol{Z})$ to mean the energy evaluated at configuration $\boldsymbol{Z}$. Consider the original mild solution $\boldsymbol{X}(t)$. The energy is monotone decreasing and is non-negative, and therefore, $\mathcal{E}(\boldsymbol{X}(t))$ converges to some value $\mathcal{E}_\star$. Since $\mathcal{E}(\boldsymbol{X}_k) = \mathcal{E}(\widehat{\boldsymbol{X}_k})$, we have $\mathcal{E}(\widehat{\boldsymbol{X}_\star}) = \mathcal{E}_\star$ (the energy is clearly continuous with respect to the $C^{1,\beta}$ norm). The same argument can be made for $\mathcal{E}(\boldsymbol{W}_\star(t)),\ 0 \leq t \leq \tau$. Indeed,

$$\mathcal{E}(\boldsymbol{W}_k(t)) = \mathcal{E}(\boldsymbol{X}(t + t_k)) \to \mathcal{E}_\star \text{ as } k \to \infty.$$

Thus, by (4.84),

$$\mathcal{E}(\boldsymbol{W}_\star(t)) = \mathcal{E}_\star,\ 0 \leq t \leq \tau.$$

From (1.24), this implies that the dissipation $\mathcal{D}$ is 0 along $\boldsymbol{W}_\star(t)$, which in turn implies that the velocity field $\boldsymbol{u}$ is identically 0. By the arguments leading to Proposition 4.3.1, this shows that that $\boldsymbol{W}_\star(t) = \widehat{\boldsymbol{X}_\star}$ is a uniformly parametrized stationary circle. We have thus shown that:

$$\left\|\widehat{\boldsymbol{X}_k} - \widehat{\boldsymbol{X}_\star}\right\|_{C^{1,\beta}} = \left\|\boldsymbol{X}_k - \boldsymbol{Z}_k\right\|_{C^{1,\beta}} \to 0 \text{ as } k \to \infty,\ \boldsymbol{Z}_k = \widehat{\boldsymbol{X}_\star} + \mathcal{P}_{\mathrm{trl}} \boldsymbol{X}_k,$$

where $\mathcal{P}_{\mathrm{trl}}$ was defined in (4.34). Note that the configurations $\boldsymbol{Z}_k$ are just the circle $\boldsymbol{X}_\star$ translated by $\mathcal{P}_{\mathrm{trl}}\boldsymbol{X}_k$. Now, note that:

$$\begin{aligned}\|\Pi \boldsymbol{X}_k\|_{C^{1,\beta}} &= \|\boldsymbol{X}_k - \mathcal{P}\boldsymbol{X}_k\|_{C^{1,\beta}} \leq \|\boldsymbol{X}_k - \boldsymbol{Z}_k\|_{C^{1,\beta}} + \|\mathcal{P}(\boldsymbol{X}_k - \boldsymbol{Z}_k)\|_{C^{1,\beta}} \\ &\leq C\,\|\boldsymbol{X}_k - \boldsymbol{Z}_k\|_{C^{1,\beta}}\,,\end{aligned}$$

where we used $\mathcal{P}\boldsymbol{Z}_k = \boldsymbol{Z}_k$ in the second inequality and the boundedness of $\mathcal{P}$ in the $C^{1,\beta}$ norm in the last inequality. Thus,

$$\lim_{k\to\infty} \|\Pi \boldsymbol{X}(t_k)\| = 0.$$

Take k sufficiently large, so that $\|\Pi \boldsymbol{X}(t_k)\|_{C^{1,\beta}}$ is small enough to apply Theorem 1.2.6 with initial data $\boldsymbol{X}(t_k)$. This concludes the proof. $\square$

Chapter 5

Fully Nonlinear Peskin Problem

We now consider the nonlinear Peskin problem as introduced in Section 1.1. Recall,

$$\partial_t \boldsymbol{X} = F(\boldsymbol{X}), \quad \boldsymbol{X}(0) = \boldsymbol{X}_0, \tag{5.1}$$

where

$$F(\boldsymbol{X}) = F_L(\boldsymbol{X}) + F_T(\boldsymbol{X}), \tag{5.2}$$

with

$$F_L(\boldsymbol{X}) := -\frac{1}{4\pi} \int_{\mathbb{S}^1} \frac{\Delta \boldsymbol{X} \cdot \partial_{\theta'} \boldsymbol{X}'}{|\Delta \boldsymbol{X}|^2} \left(T\left(\left| \partial_{\theta'} \boldsymbol{X}' \right| \right) \frac{\partial_{\theta'} \boldsymbol{X}'}{|\partial_{\theta'} \boldsymbol{X}'|} \right) d\theta' \tag{5.3}$$

and

$$F_T(\boldsymbol{X}) := -\frac{1}{4\pi} \int_{\mathbb{S}^1} \partial_{\theta'} \left(\frac{\Delta \boldsymbol{X} \otimes \Delta \boldsymbol{X}}{|\Delta \boldsymbol{X}|^2} \right) \left(T\left(\left| \partial_{\theta'} \boldsymbol{X}' \right| \right) \frac{\partial_{\theta'} \boldsymbol{X}'}{|\partial_{\theta'} \boldsymbol{X}'|} \right) d\theta'. \tag{5.4}$$

We will build a solution to (5.1) in $C([0,T]; h^{1,\gamma}) \cap C^1([0,T]; h^{\gamma})$ via application of the following theorem.

Theorem 5.0.1. ***Adapted from [21]*** *Let $E_1 \subset E_0 \subset E$ be Banach spaces and let $0 < \sigma < 1$. Given $T > 0$, open set $\mathcal{O}_1 \subset E_1$ and a function*

$$F : [0, T] \times \mathcal{O}_1 \mapsto E_0, \quad (t, x) \mapsto F(t, x)$$

such that F and F_x are continuous in $[0, T] \times \mathcal{O}_1$. If for every $(\bar{t}, \bar{u}) \in [0, T] \times \mathcal{O}_1$ we have $F_x(\bar{t}, \bar{u}) : E_1 \mapsto E_0$ is the part of a sectorial operator $A : D(A) \subset E \mapsto E$ with $D_A(\sigma) \simeq E_0$ and $D_A(\sigma + 1) \simeq E_1$ then for every $\bar{t} \in [0, T]$ and $\bar{u} \in \mathcal{O}_1$ there are $\delta > 0$, $r > 0$ such that if $t_0 \in [0, T)$, $|t_0 - \bar{t}| \leq \delta$, and $\|x_0 - \bar{u}\| \leq r$ then the problem

$$v'(t) = F(t, v(t)), \; t_0 \leq t \leq t_0 + \delta; \; v(t_0) = x_0$$

has a unique solution $v \in C([t_0, t_0 + \delta]; E_1) \cap C^1([t_0, t_0 + \delta]; E_0)$.

In order to prove local well posedness, we need only show that our problem satisfies the given hypotheses. In particular, our F is a function of $\boldsymbol{X}$ only, so, we need only think about how it acts on certain subsets of $h^{1,\gamma}$. Given $m > 0$, let us define the set

$$\mathcal{O}_m := \left\{ \boldsymbol{Y} \in h^{1,\gamma} : |\boldsymbol{Y}|_* \geq m > 0 \right\}. \tag{5.5}$$

Proposition 5.0.2. *If $m > 0$, $\gamma \in (0, 1)$ and $T \in h^{1,\gamma}$ then F as defined in* (5.2) *maps $\mathcal{O}_m \subset h^{1,\gamma}$ to h^γ.*

Proof. As defined, the kernel of F_T is of the form required by proposition 2.3.1 and therefore $F_T : O_m \subset h^{1,\gamma} \subset C^{1,\gamma} \mapsto C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor} \subset h^\gamma$. Further, F_L can be rewritten as

$$\begin{aligned} F_L(\boldsymbol{X}) = &-\frac{1}{4\pi} \mathcal{H} \left(T\left(|\partial_\theta \boldsymbol{X}|\right) \frac{\partial_\theta \boldsymbol{X}}{|\partial_\theta \boldsymbol{X}|} \right) \\ &- \frac{1}{4\pi} \int_{\mathbb{S}^1} \left(\frac{\Delta \boldsymbol{X} \cdot \partial_{\theta'} \boldsymbol{X}'}{|\Delta \boldsymbol{X}|^2} - \frac{1}{2} \cot\left(\frac{\theta - \theta'}{2} \right) \right) \left(T\left(|\partial_{\theta'} \boldsymbol{X}'|\right) \frac{\partial_{\theta'} \boldsymbol{X}'}{|\partial_{\theta'} \boldsymbol{X}'|} \right) d\theta'. \end{aligned}$$

Proposition 2.3.2 implies that the second term is in $C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor} \subset h^\gamma$ and proposition 2.3.4 implies that the first term is in $h^{0,\gamma}$. $\square$

Proposition 5.0.3. *Given any $m > 0$ and $\gamma \in (0, 1)$, if $T \in h^{1,\gamma}$ then the Gâteaux derivative of F at any $\boldsymbol{X} \in O_m$ maps $h^{1,\gamma}$ to h^γ.*

Proof. We compute the Gâteaux derivative directly. For any $\boldsymbol{X} \in h^{1,\gamma}$, consider direction $\boldsymbol{Y} \in h^{1,\gamma}$ and $\epsilon_0 > 0$ such that $|\boldsymbol{X} + \epsilon \boldsymbol{Y}|_* > 0$ for all $\epsilon < \epsilon_0$. Then, we compute

$$
\begin{aligned}
A(\boldsymbol{X})\boldsymbol{Y} &:= \frac{d}{d\epsilon} F(\boldsymbol{X} + \epsilon \boldsymbol{Y})\Big|_{\epsilon=0} \\
&= \frac{1}{4\pi} \int_{\mathbb{S}^1} \partial_{\theta'} \left(\frac{d}{d\epsilon} \left(\log |\Delta \boldsymbol{X} + \epsilon \boldsymbol{Y}| \right) \right) \left(T\left(\left| \partial_{\theta'} \boldsymbol{X}' \right| \right) \frac{\partial_{\theta'} \boldsymbol{X}'}{|\partial_{\theta'} \boldsymbol{X}'|} \right) d\theta' \Bigg|_{\epsilon=0} \\
&- \frac{1}{4\pi} \int_{\mathbb{S}^1} \partial_{\theta'} \left(\frac{d}{d\epsilon} \left(\frac{(\Delta \boldsymbol{X} + \epsilon \Delta \boldsymbol{Y}) \otimes (\Delta \boldsymbol{X} + \epsilon \boldsymbol{Y})}{|\Delta \boldsymbol{X} + \epsilon \Delta \boldsymbol{Y}|^2} \right) \right) \left(T\left(\left| \partial_{\theta'} \boldsymbol{X}' \right| \right) \frac{\partial_{\theta'} \boldsymbol{X}'}{|\partial_{\theta'} \boldsymbol{X}'|} \right) d\theta' \Bigg|_{\epsilon=0} \\
&- \frac{1}{4\pi} \int_{\mathbb{S}^1} \frac{\Delta \boldsymbol{X} \cdot \partial_{\theta'} \boldsymbol{X}'}{|\Delta \boldsymbol{X}|^2} \frac{d}{d\epsilon} \left(T\left(\left| \partial_{\theta'} \boldsymbol{X}' \right| \right) \frac{\partial_{\theta'} \boldsymbol{X}'}{|\partial_{\theta'} \boldsymbol{X}'|} \right) d\theta' \Bigg|_{\epsilon=0} \\
&- \frac{1}{4\pi} \int_{\mathbb{S}^1} \partial_{\theta'} \left(\frac{(\Delta \boldsymbol{X} + \epsilon \Delta \boldsymbol{Y}) \otimes (\Delta \boldsymbol{X} + \epsilon \boldsymbol{Y})}{|\Delta \boldsymbol{X} + \epsilon \Delta \boldsymbol{Y}|^2} \right) \frac{d}{d\epsilon} \left(T\left(\left| \partial_{\theta'} \boldsymbol{X}' \right| \right) \frac{\partial_{\theta'} \boldsymbol{X}'}{|\partial_{\theta'} \boldsymbol{X}'|} \right) d\theta' \Bigg|_{\epsilon=0} \\
&= A_1 + A_2 + A_3 + A_4.
\end{aligned}
$$

Note that lemma 2.2.8 along with proposition 2.3.1 imply that $A_1 \in C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor} \subset h^{0,\gamma}$. Similarly, lemma 2.2.6 and proposition 2.3.1 gives $A_2 \in C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor} \subset h^{0,\gamma}$ as well. Finally, we compute

$$
\begin{aligned}
&\left(T\left(\left| \partial_{\theta'} \boldsymbol{X}' \right| \right) \frac{\partial_{\theta'} \boldsymbol{X}'}{|\partial_{\theta'} \boldsymbol{X}'|} \right) \Bigg|_{\epsilon=0} \\
&\qquad = \left(\frac{T\left(|\partial_{\theta'} \boldsymbol{X}'| \right)}{|\partial_{\theta'} \boldsymbol{X}'|} + \left(T'\left(\left| \partial_{\theta'} \boldsymbol{X}' \right| \right) - \frac{T\left(|\partial_{\theta'} \boldsymbol{X}'| \right)}{|\partial_{\theta'} \boldsymbol{X}'|} \right) \frac{\partial_{\theta'} \boldsymbol{X}' \otimes \partial_{\theta'} \boldsymbol{X}'}{|\partial_{\theta'} \boldsymbol{X}'|^2} \right) \partial_{\theta'} \boldsymbol{Y}',
\end{aligned}
$$

which is in $h^{0,\gamma}$ by virtue of $|\boldsymbol{X}|_* > 0$ and products and quotients of $h^{0,\gamma}$ functions being in $h^{0,\gamma}$. Thus, we may apply proposition 2.3.1 to term A_4 to find $A_4 \in C^{\lfloor 2\gamma \rfloor, 2\gamma - \lfloor 2\gamma \rfloor} \subset h^{0,\gamma}$. Finally, as in the previous proposition, we may remove the Hilbert Transform from term A_3 and use propositions 2.3.2 and 2.3.4 to conclude that $A_3 \in h^{0,\gamma}$. □

Proposition 5.0.4. *If $T \in h^{1,\gamma}$ and both $T(s), T'(s) > 0$, then the Gâteaux derivative of F at any $\boldsymbol{X} \in O_m$ generates an analytic semigroup on the space $h^{0,\gamma}$.*

Proof. We have computed the Gâteaux derivative of F in the previous proposition. In doing so, we found that $A(\boldsymbol{X})\boldsymbol{Y} = A_1 + A_2 + A_3 + A_4$. Further, we found that each of A_1, A_2, A_4

actually mapped $h^{0,\gamma}$ to $C^{\lfloor 2\gamma\rfloor, 2\gamma-\lfloor 2\gamma\rfloor}$. Finally let us define A_3 as

$$\begin{aligned} A_3(\boldsymbol{X})\boldsymbol{Y} &= -\frac{1}{4}\mathcal{H}\left(M(\theta)\partial_\theta \boldsymbol{Y}\right) \\ &\quad - \frac{1}{4\pi}\int_{\mathbb{S}^1}\left(\frac{\Delta \boldsymbol{X}\cdot\partial_{\theta'}\boldsymbol{X}'}{|\Delta\boldsymbol{X}|^2} - \frac{1}{2}\cot\left(\frac{\theta-\theta'}{2}\right)\right)(M(\theta')\partial_{\theta'}\boldsymbol{Y}')d\theta' \\ &= A_3^1 + A_3^2, \end{aligned}$$

where

$$M(\theta) = \left(\frac{T\left(|\partial_\theta \boldsymbol{X}|\right)}{|\partial_\theta \boldsymbol{X}|} + \left(T'\left(|\partial_\theta \boldsymbol{X}|\right) - \frac{T\left(|\partial_\theta \boldsymbol{X}|\right)}{|\partial_\theta \boldsymbol{X}|}\right)\frac{\partial_\theta \boldsymbol{X}\otimes\partial_\theta \boldsymbol{X}}{|\partial_\theta \boldsymbol{X}|^2}\right).$$

Application of proposition 2.3.2 shows that the second of these terms is in $C^{\lfloor 2\gamma\rfloor, 2\gamma-\lfloor 2\gamma\rfloor}$, while application of proposition 2.3.4 shows that the first term is in $h^{0,\gamma}$. Thus, we may write operator A as

$$A = \Lambda_M + B$$

where $\Lambda_M = A_3^1$, and $B = A_1 + A_2 + A_3^2 + A_4$ is a lower order term. Provided M is symmetric positive definite, we may apply Theorem 3.2.5 and conclude that A generates an analytic semigroup on $h^{0,\gamma}$. Thus, it suffices to show that M is symmetric positive definite.

It is easy to see that M is symmetric. To show that it is positive definite, Let us rewrite M as

$$\begin{aligned} M &= M_1 + M_2 \\ M_1 &:= \frac{T'(|\partial_\theta \boldsymbol{X}|)}{|\partial_\theta \boldsymbol{X}|^2}\partial_\theta \boldsymbol{X}\otimes\partial_\theta \boldsymbol{X} \\ M_2 &:= \frac{T(|\partial_\theta \boldsymbol{X}|)}{|\partial_\theta \boldsymbol{X}|^3}\left(|\partial_\theta \boldsymbol{X}|^2\boldsymbol{I} - \partial_\theta \boldsymbol{X}\otimes\partial_\theta \boldsymbol{X}\right). \end{aligned} \tag{5.6}$$

We first consider M_1 and M_2 separately. Note that M_1 and M_2 are both positive semi-definite as $T, T' > 0$ implies $\mathrm{Tr}(M_1), \mathrm{Tr}(M_2) > 0$ and $\det M_1 = \det M_2 = 0$. Hence, as the sum of two positive semi-definite matrices, M is also positive semi-definite. To show that it is positive definite, note that $M_1\boldsymbol{V} = \boldsymbol{0}$ if and only if $\boldsymbol{V}\cdot\partial_\theta \boldsymbol{X} = 0$. If $\boldsymbol{V}\cdot\partial_\theta \boldsymbol{X} = 0$, then $\boldsymbol{V} = \alpha(-\partial_\theta Y, \partial_\theta X)^T$ for some $\alpha\in\mathbb{R}$ with $\alpha\neq 0$. Applying M_2 to $\boldsymbol{V}$ gives

$$
\begin{aligned}
M_2\boldsymbol{V} &= \frac{T(|\partial_\theta \boldsymbol{X}|)}{|\partial_\theta \boldsymbol{X}|^3}\left(|\partial_\theta \boldsymbol{X}|^2\,\boldsymbol{I} - \partial_\theta \boldsymbol{X}\otimes\partial_\theta \boldsymbol{X}\right)\boldsymbol{V} \\
&= \frac{T(|\partial_\theta \boldsymbol{X}|)}{|\partial_\theta \boldsymbol{X}|}\boldsymbol{V}.
\end{aligned}
$$

Thus,

$$
\boldsymbol{V}^T M\boldsymbol{V} = \frac{T(|\partial_\theta \boldsymbol{Z}|)}{|\partial_\theta \boldsymbol{Z}|}\,|\boldsymbol{V}|^2 > 0.
$$

Finally, if $\boldsymbol{V}\cdot\partial_\theta \boldsymbol{X} \neq 0$ then

$$
\boldsymbol{V}^T M\boldsymbol{V} = \boldsymbol{V}^T M_1\boldsymbol{V} + \boldsymbol{V}^T M_2\boldsymbol{V} \geq \boldsymbol{V}^T M_1\boldsymbol{V} > 0,
$$

showing that M is positive definite. $\square$

We now have all of the necessary pieces to prove Theorem 1.2.9.

Proof of Theorem 1.2.9. Let $\boldsymbol{X}_0 \in h^{1,\gamma}$ with $|\boldsymbol{X}_0|_*$ be the initial configuration of the system. Then, define A as the linearization of F about $\boldsymbol{X}_0$. Choose any $0 < \alpha < \gamma$. Then, propositions 5.0.2 and 5.0.3 imply that $F : O_m \subset h^{1,\gamma} \mapsto h^{0,\gamma} \subset h^{0,\alpha}$. Thus, proposition 5.0.4 states that A generates an analytic semigroup on $h^{0,\gamma}$. Further, proposition 3.2.7 implies that there is some $\sigma \in (0,1)$ such that $h^{1,\gamma} = D_A(\sigma+1)$ and $h^{0,\gamma} = D_A(\sigma)$. Thus an application of Theorem 5.0.1 yields the desired result. $\square$

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